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Analytical Solution for Intensive Quenching of Cylindrical Sample

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Abstract

In this paper we construct exact solution for cylindrical wall with fin. We assume the heat transfer process for hyperbolic heat equation in cylindrical sample.

Introduction

Usually mathematical modelling of systems with extended surfaces is realized by one dimensional steady-state assumptions [1]. In our previous papers we have constructed two and three dimensional analytical approximate [2]-[5] solutions for the steady-state process.

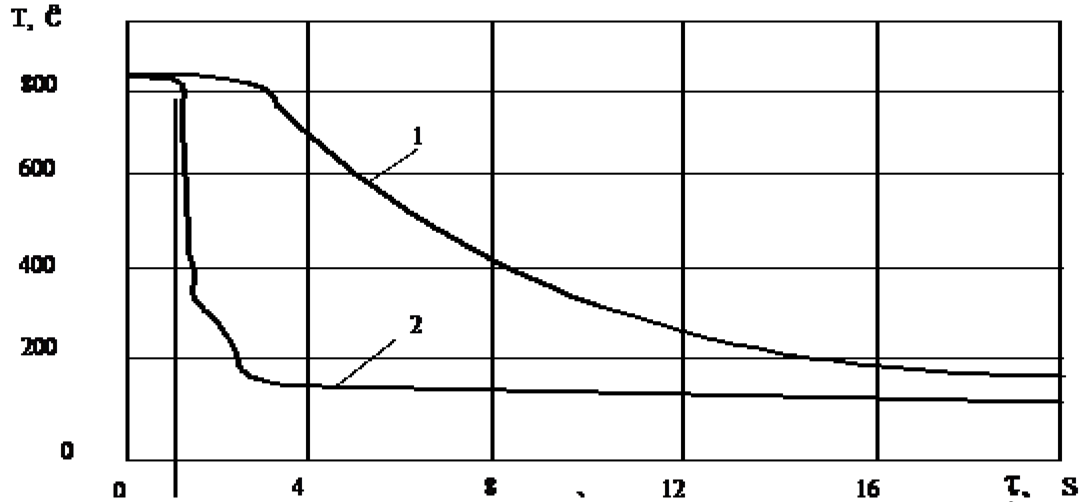

Fig. 1. Function of temperature at cylinder-shaped sample at its quenching from 860°C in water solution of $CaCl_2$: 1 – centre; 2 – surface

New method for steel quenching is intensive steel quenching in the salt water [6], [7]. One of authors has idea to describe the steel cooling process by hyperbolic heat equation [8], [9], including rectangular samples with fin. Thanks to Doctor N. Kobasko we can see experimental results in figure 1. Curve Nr. 2 on this figure shows that solution has character of equation of hyperbolic type. As concluded this hyperbolic type heat exchange equation better describes steel quenching physical processes. It is necessary to know temperature's initial distribution and derivation by time at the beginning of process. We can not gain such data by experimental way. But it is possible to get experimentally temperature change during and after quenching.

Here we construct non steady-state solution for hyperbolic heat equation for cylindrical sample. This way gives more suitable form of the solution in the form of Fredholm integral equation. We reduce exact 3-D problem to two dimensional and obtain exact analytical solutions by the Green function method.

1. Mathematical Formulation of 3-D Problem

We will consider cylindrical wall and fin. Let's assume that surface $z = 0$ and $\varphi = 0$ is middle surfaces with second type homogeneous symmetry boundary conditions. In other words our sample angle width is 2Φ and height in *z* direction is 2*H.* All other surfaces come to contact with continuously flowing water at constant temperature Θ_0. We assume that heat exchange processes on surfaces can be described by linear third type boundary conditions.

Steel sample is heated up to beginning temperature V_{in} and placed in facility for quenching. This physical system we will describe with following mathematical model.

We will start with accurate three-dimensional formulation of transient problem for system of cylindrical wall and fin. The one element of the wall (base) is placed in the non-dimensional domain $r \in \; r_0, r_1 \;, z \in \; 0, H \;, \varphi \in \; 0, \Phi$. The cylindrical fin occupies the domain $r \in \; r_1, r_2 \;, z \in \; 0, H_0 \;, \varphi \in \; 0, \Phi$.

We will use following dimensionless arguments, parameters to transform our problem to dimensionless problem: $r = \frac{\tilde{r}}{H}, z = \frac{\tilde{z}}{H}, \rho_0 = \frac{r_0}{H}, \quad \rho_1 = \frac{r_1}{H}, \rho_2 = \frac{r_2}{H}, \quad b = \frac{H_0}{H}, \beta = \frac{hH}{k},$ $\gamma = \frac{\beta}{\rho_1}, a^2 = \frac{k}{c\rho}$. Here k - heat conductivity coefficient for the fin and wall, c - is specific heat capacity, ρ - density, h - heat exchange coefficient for the system, H_0 - width (thickness) of the fin, L - length of the fin, H - thickness of the wall, parameter τ is so called relaxation time. One element of the wall (base) placed in the domain now is $r \in \; \rho_0, \rho_1 \;, z \in \; 0,1 \;, \varphi \in \; 0, \Phi$.

Let's discuss most interesting shapes of our system. Case when $r_0 = \rho_0 = 0$ there are two shape options: a) $\Phi < 2\pi$ partly cylinder (cross section in z direction is sector) with fin; b) $\Phi = 2\pi$ complete cylinder with fin. Case when $r_0 > 0$ and $\Phi = 2\pi$ we get tube with fin. Let us consider an interesting and peculiar version: complete cylinder with pin orthogonal to the (side) profile surface. On the supposition that on the side surface of the cylinder at point with coordinates z_0, φ_0 is the centre line of the pin. The radius of the pin is ρ_3, height h_3. This kind of geometry is also allowed by our method.

We describe the dimensional temperature field of steel sample (wall) by function $\bar{V}_0(r, z, \varphi, t)$ and introduce following dimensionless temperature field. Here V_{in} is some characteristic value:

$$V_0(r, z, \varphi, t) = \frac{\bar{V}_0(r, z, \varphi, t) - \Theta_0}{V_{in} - \Theta_0}.$$

It means, that we describe dimensionless temperature field $V_0(r, z, \varphi, t)$ in the wall with the equation:

$$\tau \frac{\partial^2 V_0}{\partial t^2} + \frac{\partial V_0}{\partial t} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V_0}{\partial r} \right) + a^2 \frac{\partial^2 V_0}{\partial z^2} + \frac{a^2}{r^2}\frac{\partial V_0}{\partial \varphi^2}. \tag{1.1}$$

Similarly the temperature field $V(r, z, \varphi, t)$ in the fin fulfils the equation:

$$\tau \frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r} \right) + a^2 \frac{\partial^2 V}{\partial z^2} + \frac{a^2}{r^2}\frac{\partial V}{\partial \varphi^2}. \tag{1.2}$$

And we have following boundary conditions in φ and z directions:

$$\left.\frac{\partial V_0}{\partial \varphi}\right|_{\varphi=0}=0,\left.\left(\frac{\partial V_0}{\partial \varphi}+\beta V_0\right)\right|_{\varphi=\Phi}=0,\left.\frac{\partial V}{\partial \varphi}\right|_{\varphi=0}=0,\left.\left(\frac{\partial V}{\partial \varphi}+\beta V\right)\right|_{\varphi=\Phi}=0,$$
$$\left.\frac{\partial V_0}{\partial z}\right|_{z=0}=0,\left.\left(\frac{\partial V_0}{\partial z}+\beta V_0\right)\right|_{z=1}=0,\left.\frac{\partial V}{\partial z}\right|_{z=0}=0,\left.\left(\frac{\partial V}{\partial z}+\beta V\right)\right|_{z=b}=0. \tag{1.3}$$

Analogously we define the boundary conditions in r direction (one symmetry condition and three heat exchange conditions):

$$\left.\left(\frac{\partial V_0}{\partial r}-\beta V_0\right)\right|_{r=\rho_0}=0,\left.\left(\frac{\partial V_0}{\partial r}+\beta V_0\right)\right|_{r=\rho_1}=0,\left.\frac{\partial V}{\partial r}\right|_{z=0}=0,\left.\left(\frac{\partial V}{\partial r}+\beta V\right)\right|_{r=\rho_2}=0. \tag{1.4}$$

Of course, we can assume the conjugations conditions on the surface between the wall and the fin as ideal thermal contact - there is no contact resistance (is continuity of temperature and the heat flux):

$$U_0\big|_{\rho_1-0}=U\big|_{\rho_1+0},\ \left.\frac{\partial U_0}{\partial r}\right|_{r=\rho_1-0}=\left.\frac{\partial U}{\partial r}\right|_{r=\rho_1+0},y\in[0,b]. \tag{1.5}$$

The initial conditions are assumed in following form:

$$V_0\big|_{t=0}=V_0^0(r,z,\varphi),V\big|_{t=0}=V_0(r,z,\varphi), \tag{1.6}$$

$$\left.\frac{\partial V_0}{\partial t}\right|_{t=0}=\tilde{W}_0^0(r,z,\varphi),\left.\frac{\partial V}{\partial t}\right|_{t=0}=\tilde{W}_0(r,z,\varphi). \tag{1.7}$$

From the practical point of view the both conditions (1.7) are unrealistic. For this case the initial heat flux must be determined theoretically. As additional condition we assume that the temperature distribution and the heat fluxes distribution at the end of process are given (known):

$$V_0\big|_{t=T}=V_T^0(r,z,\varphi),V\big|_{t=T}=V_T(r,z,\varphi),\left.\frac{\partial V_0}{\partial t}\right|_{t=T}=\tilde{W}_T^0(r,z,\varphi),\left.\frac{\partial V}{\partial t}\right|_{t=T}=\tilde{W}_T(r,z,\varphi). \tag{1.8}$$

In case of $\Phi=2\pi$, boundary condition (1.3) is in following form:

$$\left.\frac{\partial V_0}{\partial \varphi}\right|_{\varphi=0}=\left.\frac{\partial V_0}{\partial \varphi}\right|_{\varphi=\Phi},\left.\frac{\partial V}{\partial \varphi}\right|_{\varphi=0}=\left.\frac{\partial V}{\partial \varphi}\right|_{\varphi=\Phi}. \tag{1.9}$$

2. Exact 3-D Reduction to 2-D Problem

Introducing following average integral values for argument φ we can reduce equations (1.1) and (1.2) from 3-D to 2-D problem:

$$U\ r,z,t\ =\frac{1}{\Phi}\int_0^{\Phi}V(r,z,\varphi,t)d\varphi,U_0\ r,z,t\ =\frac{1}{\Phi}\int_0^{\Phi}V_0(r,z,\varphi,t)d\varphi. \tag{2.1}$$

Integration of the equation (1.1) for the wall over $\varphi\in[0,\Phi]$ gives following equation (exact consequence of 3-D partial differential equation (1.1)):

$$\tau\frac{\partial^2 U_0}{\partial t^2}+\frac{\partial U_0}{\partial t}=\frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_0}{\partial r}\right)+a^2\frac{\partial^2 U_0}{\partial z^2}+\frac{a^2}{r^2\Phi}\left(\left.\frac{\partial V_0}{\partial \varphi}\right|_{\varphi=\Phi}-\left.\frac{\partial V_0}{\partial \varphi}\right|_{\varphi=0}\right).$$

The first pair of boundary conditions (1.3) allows rewriting the last equality in form of two dimensional equations (assuming U_0 is constant regarding argument φ):

$$\tau\frac{\partial^2 U_0}{\partial t^2}+\frac{\partial U_0}{\partial t}=\frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_0}{\partial r}\right)+a^2\frac{\partial^2 U_0}{\partial z^2}-d(r)U_0,d(r)=\frac{\beta a^2}{r^2\Phi}. \tag{2.2}$$

Similarly we can describe the dimensionless temperature field $U(r,z,t)$ in the fin with the equation:

$$\tau \frac{\partial^2 U}{\partial t^2} + \frac{\partial U}{\partial t} = \frac{a^2}{r} \frac{\partial}{\partial r}\left(r \frac{\partial U}{\partial r} \right) + a^2 \frac{\partial^2 U}{\partial z^2} - d(r)U \ . \qquad (2.3)$$

We add needed boundary conditions and conjugations conditions as follow:

$$\frac{\partial U_0}{\partial r} - \beta U_0 = 0 \, , r = \rho_0 \, , z \in [0,1] \, , \qquad (2.4)$$

$$\frac{\partial U_0}{\partial r} + \beta U_0 = 0 \, , r = \rho_1 \, , \ z \in [b,1] \, , \qquad (2.5)$$

$$\left. \left(\frac{\partial U_0}{\partial z} + \beta U_0 \right) \right|_{z=1} = 0, r \in [\rho_0, \rho_1], \qquad (2.6)$$

$$\frac{\partial U_0}{\partial z} = 0, r = [\rho_0, \rho_1], z = 0, \qquad (2.7)$$

$$\frac{\partial U}{\partial r} + \beta U = 0 \, , \ r = \rho_2 \, , \ z \in [0,b] \, . \qquad (2.8)$$

$$\frac{\partial U}{\partial z} = 0, r = [\rho_1, \rho_2], z = 0, \qquad (2.9)$$

$$\frac{\partial U}{\partial z} + \beta U = 0 \, , \ r = [\rho_1, \rho_2] \, , \ z = b \, , \qquad (2.10)$$

$$U_0 \big|_{\rho_1 - 0} = U \big|_{\rho_1 + 0} \, , \ \left. \frac{\partial U_0}{\partial r} \right|_{\rho_1 - 0} = \left. \frac{\partial U}{\partial r} \right|_{\rho_1 + 0} . \qquad (2.11)$$

The initial conditions are transformed in form:

$$U_0 \big|_{t=0} = U_0^0(r,z), U \big|_{t=0} = U_0(r,z), \qquad (2.12)$$

$$\left. \frac{\partial U_0}{\partial t} \right|_{t=0} = W_0^0(r,z), \left. \frac{\partial U}{\partial t} \right|_{t=0} = W_0(r,z). \qquad (2.13)$$

Additional conditions transform so:

$$U_0 \big|_{t=T} = U_T^0(r,z), U \big|_{t=T} = U_T(r,z), \left. \frac{\partial V_0}{\partial t} \right|_{t=T} = W_T^0(r,z), \left. \frac{\partial V}{\partial t} \right|_{t=T} = W_T(r,z). \qquad (2.14)$$

All initial conditions are gained by integration of conditions (1.6) – (1.8) by direction φ regarding equation (2.1). Equations (1.1), (1.2) in case of $\Phi = 2\pi$ and boundary condition (1.9) are rewritten in more simple form:

$$\begin{aligned} &\tau \frac{\partial^2 U_0}{\partial t^2} + \frac{\partial U_0}{\partial t} = \frac{a^2}{r} \frac{\partial}{\partial r}\left(r \frac{\partial U_0}{\partial r} \right) + a^2 \frac{\partial^2 U_0}{\partial z^2} \, , \\ &\tau \frac{\partial^2 U}{\partial t^2} + \frac{\partial U}{\partial t} = \frac{a^2}{r} \frac{\partial}{\partial r}\left(r \frac{\partial U}{\partial r} \right) + a^2 \frac{\partial^2 U}{\partial z^2} . \end{aligned} \qquad (2.15)$$

Let's make following transformation to functions $v(r,z,t)$, $v_0(r,z,t)$:

$$U(r,z,t) = \exp\left(-\frac{t}{2\tau} \right) v(r,z,t), \ U_0(r,z,t) = \exp\left(-\frac{t}{2\tau} \right) v_0(r,z,t) . \qquad (2.16)$$

Main equations (2.2) and (2.3) transforms in following form:

$$\frac{\partial^2 v_0}{\partial t^2} = \frac{a_\tau^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_0}{\partial r}\right) + a_\tau^2\frac{\partial^2 v_0}{\partial z^2} - cv_0,\ a_\tau^2 = \frac{a^2}{\tau},$$
$$\frac{\partial^2 v}{\partial t^2} = \frac{a_\tau^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + a_\tau^2\frac{\partial^2 v}{\partial z^2} - cv, c = -\frac{1}{4\tau^2}. \quad (2.17)$$

Boundary conditions (2.5) - (2.11) for functions $v(r,z,t),\ v_0(r,z,t)$ stay in same form, only initial conditions are following:

$$v_0\big|_{t=0} = U_0^0(r,z), v\big|_{t=0} = U_0(r,z), \quad (2.18)$$

$$\left.\frac{\partial v_0}{\partial t}\right|_{t=0} = W_0^0(r,z) + \frac{U_0^0(r,z)}{2\tau}, \left.\frac{\partial v}{\partial t}\right|_{t=0} = W_0(r,z) + \frac{U_0(r,z)}{2\tau}. \quad (2.19)$$

Additional conditions (2.14) are the following:

$$v_0\big|_{t=T} = \exp\left(\frac{T}{2\tau}\right)U_T^0(r,z),\ \left.\frac{\partial v_0}{\partial t}\right|_{t=T} = \exp\left(\frac{T}{2\tau}\right)\left[W_T^0(r,z) + \frac{U_T^0(r,z)}{2\tau}\right],$$
$$v\big|_{t=T} = \exp\left(\frac{T}{2\tau}\right)U_T(r,z), \left.\frac{\partial v}{\partial t}\right|_{t=T} = \exp\left(\frac{T}{2\tau}\right)\left[W_T(r,z) + \frac{U_T(r,z)}{2\tau}\right]. \quad (2.20)$$

3. Solution of 2-D Problem

We can not discuss all mentioned cases by lack of place, but almost all cases have similar methodology of research as in papers [8], [9]. We will consider complete cylinder with fin. In this case we prefer to split up our sample in two complete cylinders connected with surface $z = b$:

$$V_0\big|_{b-0} = V\big|_{b+0}, \left.\frac{\partial V_0}{\partial z}\right|_{z=b-0} = \left.\frac{\partial V}{\partial z}\right|_{z=b+0}, r \in [0, \rho_2]. \quad (3.1)$$

We rewrite the boundary condition for cylinder on the right hand side (2.5) together with the conjugation conditions (3.1) in following common form:

$$\left.\left(\frac{\partial V_0}{\partial z} + \beta V_0\right)\right|_{z=b-0} = \begin{cases} F_0(r,t), 0 < r < \rho_2, \\ 0 \quad\ , \rho_2 < r < \rho_1, \end{cases} F_0(r,t) = \left.\left(\frac{\partial V}{\partial z} + \beta V\right)\right|_{z=b+0}. \quad (3.2)$$

The solution for the complete cylinder can be written in well known form by means of Green function, see, [10]:

$$V_0(r,z,t) = \Phi_0(r,z,t) + a_\tau^2\int_0^t d\tau_0\int_0^{\rho_2} F_0(\rho_0,\tau_0)G_0(r,z,\rho_0,b,t-\tau_0)d\rho_0,$$
$$\Phi_0(r,z,t) = \int_0^{\rho_1} d\xi_0\int_0^1\left[\frac{1+\tau}{\tau}U_0^0(\xi_0,\eta_0) + W_0^0(\xi_0,\eta_0)\right]G_0(r,z,\xi_0,\eta_0,t)d\eta_0. \quad (3.3)$$

The Green function has the form:

$$G_0(r,z,\xi,\eta,t) = \frac{2\xi}{\rho_1^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\mu_n^2}{\beta^2\rho_1^2 + \mu_n^2}J_0\left(\frac{\mu_n r}{\rho_1}\right)J_0\left(\frac{\mu_n\xi}{\rho_1}\right)\frac{\varphi_m(z)\varphi_m(\eta)}{\|\varphi_m\|^2}\frac{\sin(t\sqrt{\lambda_{mn}})}{\sqrt{\lambda_{mn}}},$$

$$\varphi_m(z) = \cos(\beta_m z), \lambda_{nm} = a_\tau^2\left(\frac{\mu_n^2}{\rho_1^2} + \beta_m^2\right) - \frac{1}{4\tau^2}, \|\varphi_m\|^2 = \frac{b}{2} + \frac{\beta}{2}\left(1 + \frac{\beta_m^2}{\beta_m^2 + \beta^2}\right).$$

Here $\lambda_m(\mu_n)$ are positive roots of following transcendental equation:

$$\beta_m = \frac{(m-1)\pi + \pi/4}{b}, m = \overline{1,\infty}; \frac{J_1(\mu)}{J_0(\mu)} = \frac{\beta\rho_1}{\mu}.$$

The representation is not the solution because of the unknown function $F_0(\eta_0, \tau_0)$.
Now we look to the fin and rewrite the conjugations conditions in following form:

$$\left(\frac{\partial V}{\partial z} - \beta V \right)\Bigg|_{z=b+0} = \begin{cases} F \;\; r,t \;\; , 0 < r < \rho_2, \\ 0, \quad \rho_2 < r < \rho_1, \end{cases} F(r,t) = \left(\frac{\partial V_0}{\partial z} - \beta V_0 \right)\Bigg|_{z=b-0}. \tag{3.4}$$

Similar as in case of formula (4.3) the solution for the fin can be represent in following form:

$$V(r,z,t) = \Phi(r,z,t) + a_\tau^2 \int_0^t d\tau \int_0^{\rho_2} F(\rho,\tau) G_0(r,z,\rho,b,t-\tau) d\rho. \tag{3.5}$$

Formulae (3.3), (3.5) can be rewritten for $F_0(r,t), F \;\; r,t$ and we have two Volterra-Fredholm integral equations. As in papers [8], [9] we can reduce this system to one integral equation.

Conclusions

We have constructed exact two-dimensional analytical solution for a system with cylindrical fin when the wall and the fin consist of the same material - steel.

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References

[1] Kern, D. Q., Kraus, A. D.: *Extended Surface Heat Transfer.* McGraw-Hill Book Company, 1972.

[2] Buikis, A., Buike, M.: *Closed two-dimensional solution for heat transfer in a periodical system with a fin*. Proceedings of the Latvian Academy of Sciences, Section B, Vol. 52, Nr. 5, 1998, pp. 218-222.

[3] Buikis, A., Buike, M.: *Some analytical 3-D steady-state solutions for systems with rectangular fin*. IASME Transactions. Issue 7, Vol. 2, 2005, pp. 1112-1119.

[4] Bruvere, A., Buikis, A.: *Some approximate analytical steady-state solutions for cylindrical fin*. Proceedings of 4th IASME/WSEAS International Conference on Heat Transfer, Thermal Engineering and Environment, 2006, pp. 238-243.

[5] Bruvere, A., Buikis, A.: *Analytical 3-D steady-state statement for cylindrical fin and some of it's approximate solutions*. WSEAS Transactions on Heat and Mass Transfer. Issue 4, Vol. 1, 2006, p. 415-422.

[6] Kobasko, N. I.: *Steel Quenching in Liquid Media under Pressure*. Kyiv, Naukova Dumka, 1980.

[7] Aronov, M. A., Kobasko, N. I., Powell, J. A.: *Intensive Quenching of Carburized Steel Parts*. IASME Transactions, Issue 9, Vol. 2, November 2005, p. 1841-1845.

[8] Buike, M., Buikis, A.:*Hyperbolic Heat Equation as Mathematical Model for Steel Quenching of L-Shape Samples, Part 1 (Direct Problem)*. Proceedings of 13th WSEAS Int. Conf. on Applied Mathematics (MATH'08), WSEAS Press, 2008, pp. 198-203.

[9] Bobinska, T., Buike, M., Buikis, A.: *Hyperbolic Heat Equation as Mathematical Model for Steel Quenching of L-Shape Samples, Part 2.* (Inverse Problem), Proceedings of the 5th IASME/WSEAS Int. Conf. on Continuum Mechanics, Fluids, Heat. Cambridge, UK, WSEAS Press, 2010, pp. 21-26.

[10] Polyanin, A. D.: *Handbook of Linear Partial Differential Equations for Engineers and Scientists.* Chapman&Hall/CRC, 2000.

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