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Analytical Two - Dimensional Solution for Transient Process in the System with Rectangular Fins

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Abstract

In the paper we construct exact analytical three-dimensional solution for transient heat transfer in a double-fin assembly with two fines attached on either sides of an isolated wall. Heat exchange takes place at non-homogeneous environment.

Introduction

Extended surface is used specially to enhance the heat transfer between a solid and surrounding medium. Such an extended surface is termed a fin. The rate of heat transfer is directly proportional to the extent of the wall surface, the heat transfer coefficient and to the temperature difference between solid and the surrounding medium. Finned surfaces are widely used in many applications such as air conditioners, aircrafts, chemical processing plants, etc. Finned surfaces are also used in cooling electronic components.

In [1] is considered performance of a heat exchanger consisting of rectangular fins attached to both sides of plane wall. In [1] work one-dimensional steady-state double-fin assembly problem is compared with the single-fin assembly. In paper [2] mathematical three-dimensional formulation of transient problem for one element with one rectangular fin is examined, reduce it by conservative averaging method [3] to the system of three heat equations. Such transient problem for heat equation is interesting for intensive steel quenching [4].

1. Mathematical Formulation of 3-D Problem

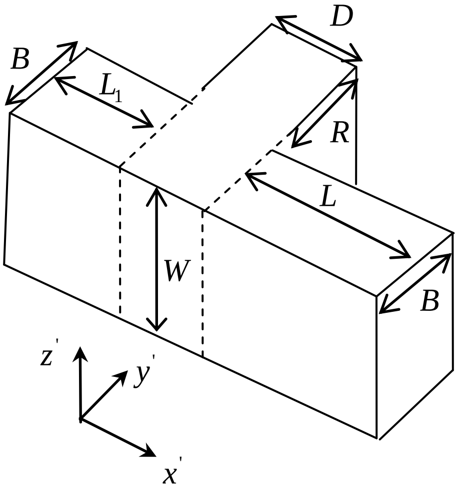

Fig. 1. One element of system with two rectangular fins

In this part we will consider full mathematical three-dimensional formulation of transient problem for one element of system with two rectangular fins. We will use following dimensionless arguments, parameters

$$x = \frac{x'}{B+R},\ y = \frac{y'}{B+R},\ z = \frac{z'}{B+R},$$

$$l = \frac{L}{B+R}, l_1 = \frac{L_1}{B+R}, w = \frac{W}{B+R},$$

$$b = \frac{B}{B+R}, \delta = \frac{D}{B+R},$$

$$\beta_{11} = \frac{h_0 \cdot (B+R)}{k_0}, \beta_{13} = \frac{h_0 \cdot (B+R)}{k_1}, \beta_{21} = \frac{h \cdot (B+R)}{k_0}, \beta_{22} = \frac{h \cdot (B+R)}{k},$$

and temperatures:

$$\bar{V}(x,y,z,t) = \frac{\tilde{V}(x,y,z,t) - T_a(t)}{T_b(t) - T_a(t)}, \bar{V}_0(x,y,z,t) = \frac{\tilde{V}_0(x,y,z,t) - T_a(t)}{T_b(t) - T_a(t)},$$

$$\bar{V}_1(x,y,z,t) = \frac{\tilde{V}_1(x,y,z,t) - T_a(t)}{T_b(t) - T_a(t)},$$

$$\bar{\Theta}(x,y,z,t) = \frac{\tilde{\Theta}(x,y,z,t) - T_a(t)}{T_b(t) - T_a(t)}, \ \bar{\Theta}_0(x,y,z,t) = \frac{\tilde{\Theta}_0(x.y,z,t) - T_a(t)}{T_b(t) - T_a(t)}.$$

We have introduced following dimensional thermal and geometrical parameters: k_0, k, k_1 - heat conductivity coefficients for the wall, right fin and left fin, $h(h_0)$ - heat exchange coefficient for the fin (wall), $2B$ – fin width (thickness), L – right fin length, L_1 – left fin length, D- thickness of the wall, W – walls' width (length), $2R$ – distance between two fins (fin spacing). Further, $\tilde{\Theta}_0(x.y,z,t)$ is the surrounding (environment) temperature on the left (hot) side (the heat source side) of the wall, $\tilde{\Theta}(x,y,z,t)$ - the surrounding temperature on the right (cold - the heat sink side) of the wall and the fin. Finally, $\tilde{V}_0(x,y,z,t), \tilde{V}(x,y,z,t), \tilde{V}_1(x,y,z,t)$ are the dimensional temperatures in the wall, right fin and left fin where $T_a(T_b)$ are integral averaged environment temperatures over appropriate edges. The one element of the wall (base) is placed in the domain $\{x \in [0,\delta], y \in [0,1], z \in [0,w]\}$. The rectangular right fin in dimensionless arguments occupies the domain $\{x \in [\delta, \delta + l], y \in [0,b], z \in [0,w]\}$. The rectangular left fin in dimensionless arguments occupies the domain $\{x \in [-l_1, 0], y \in [0,b], z \in [0,w]\}$.

We describe the temperature field by functions $\bar{V}(x,y,z,t), \bar{V}_0(x,y,z,t), \bar{V}_1(x,y,z,t)$ in the wall and fins

$$\begin{cases} \dfrac{\partial^2 \bar{V}_0}{\partial x^2} + \dfrac{\partial^2 \bar{V}_0}{\partial y^2} + \dfrac{\partial^2 \bar{V}_0}{\partial z^2} = \dfrac{1}{a_0^2} \dfrac{\partial \bar{V}_0}{\partial t}, \\ \dfrac{\partial^2 \bar{V}}{\partial x^2} + \dfrac{\partial^2 \bar{V}}{\partial y^2} + \dfrac{\partial^2 \bar{V}}{\partial z^2} = \dfrac{1}{a^2} \dfrac{\partial \bar{V}}{\partial t}, \\ \dfrac{\partial^2 \bar{V}_1}{\partial x^2} + \dfrac{\partial^2 \bar{V}_1}{\partial y^2} + \dfrac{\partial^2 \bar{V}_1}{\partial z^2} = \dfrac{1}{a_1^2} \dfrac{\partial \bar{V}_1}{\partial t}. \end{cases}$$

We assume heat fluxes from the flank surfaces (edges) and from the top and the bottom edges

$$\left.\frac{\partial \bar{V}_0}{\partial z}\right|_{z=0} = Q_{0,2}(x,y,t),\ \left.\frac{\partial \bar{V}_0}{\partial z}\right|_{z=w} = Q_{0,3}(x,y,t),\ \left.\frac{\partial \bar{V}}{\partial z}\right|_{z=0} = Q_2(x,y,t),$$
$$\left.\frac{\partial \bar{V}}{\partial z}\right|_{z=w} = Q_3(x,y,t),\ \left.\frac{\partial \bar{V}_1}{\partial z}\right|_{z=0} = Q_{1,2}(x,y,t),\ \frac{\partial \bar{V}_1}{\partial z}_{z=w} = Q_{1,3}(x,y,t). \tag{1}$$

Such type of boundary conditions (BC) allows us to make the exact reducing of this three-dimensional problem to two-dimensional problem by conservative averaging method [3]. Realizing the integration of main equation by usage of the BC (1) we obtain

$$\begin{cases} \frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} + Q_0(x,y,t) = \frac{1}{a_0^2}\frac{\partial V_0}{\partial t}, \\ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + Q(x,y,t) = \frac{1}{a^2}\frac{\partial V}{\partial t}, \\ \frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + Q_1(x,y,t) = \frac{1}{a_1^2}\frac{\partial V_1}{\partial t}. \end{cases} \tag{2}$$

We add to main partial differential equations (2) needed BC as follow

$$\left.\left(\frac{\partial V_0}{\partial x} + \beta_{11}[\vartheta_0(x,y,t) - V_0]\right)\right|_{x=0} = 0, y \in (b,1),$$
$$\left.\left(\frac{\partial V_0}{\partial x} + \beta_{21}[V_0 - \vartheta(x,y,t)]\right)\right|_{x=\delta} = 0, y \in (b,1),$$
$$\left.\frac{\partial V_0}{\partial y}\right|_{y=0} = Q_{0,0}(x,t), x \in (0,\delta),$$
$$\left.\frac{\partial V_0}{\partial y}\right|_{y=1} = Q_{0,1}(x,t), x \in (0,\delta). \tag{3}$$

We assume them as ideal thermal contact between wall and fins- there is no contact resistance

$$V_1|_{x=0-0} = V_0|_{x=0+0},$$
$$\beta_{11}\left.\frac{\partial V_1}{\partial x}\right|_{x=0-0} = \beta_{13}\left.\frac{\partial V_0}{\partial x}\right|_{x=0+0}. \tag{4}$$

$$V_0|_{x=\delta-0} = V|_{x=\delta+0},$$
$$\beta_{22}\left.\frac{\partial V_0}{\partial x}\right|_{x=\delta-0} = \beta_{21}\left.\frac{\partial V}{\partial x}\right|_{x=\delta+0}. \tag{5}$$

We have following BC for the right fin

$$\left.\left(\frac{\partial V}{\partial x}+\beta_{22}[V-\vartheta(x,y,t)]\right)\right|_{x=\delta+l}=0,\ y\in(0,b),$$
$$\left.\left(\frac{\partial V}{\partial y}+\beta_{22}[V-\vartheta(x,y,t)]\right)\right|_{y=b}=0,\ x\in(\delta,\delta+l), \tag{6}$$
$$\left.\frac{\partial V}{\partial y}\right|_{y=0}=Q_0(x,t),\ x\in(\delta,\delta+l).$$

We have following BC for the left fin

$$\left.\left(\frac{\partial V_1}{\partial x}+\beta_{13}[\vartheta_0(x,y,t)-V_1]\right)\right|_{x=-l_1}=0,\ y\in(0,b),$$
$$\left.\left(\frac{\partial V_1}{\partial y}+\beta_{13}[V_1-\vartheta_0(x,y,t)]\right)\right|_{y=b}=0,\ x\in(-l_1,0), \tag{7}$$
$$\left.\frac{\partial V_1}{\partial y}\right|_{y=0}=Q_{1,0}(x,t),\ x\in(-l_1,0).$$

Finally, we must add initial conditions for the heat equations (2)

$$V_0|_{t=0}=V_0^0(x,y),\qquad V|_{t=0}=V^0(x,y),\qquad V_1|_{t=0}=V_1^0(x,y). \tag{8}$$

2. Exact Solution of the Simplified 2-D Problem

We would like to explain the main idea of solution for the 2-D case of periodical system with constant dimensionless environmental temperatures $\vartheta_0=1(\Theta_0=T_b)$ and $\vartheta=0\,(\Theta=T_a)$. We neglect additionally the heat fluxes from flank edges. In this particular case we have $Q_0(x,y,t)=Q(x,y,t)=Q_1(x,y,t)=0$ in (2). We consider $U(x,y,t)$ is the temperature of the right fin, $U_0(x,y,t)$ temperature of the wall and $U_1(x,y,t)$ is the temperature of the left fin. The BC (1) is assumed to be homogeneous. Fluxes in BC (3), (6) and (7) are also homogeneous. Initial conditions are still standing in the form (8).

The conjugations conditions on the line between the wall and the left fin are still standing in the form (4) for the functions $U_0(x,y,t)$ and $U_1(x,y,t)$. The linear combination of the equations (4) together with first BC from (3) allow us rewrite them as following BC on the left hand side of the wall:

$$\left.\left(\frac{\partial U_0}{\partial x}-\beta_{11}U_0\right)\right|_{x=0+0}=\beta_{11}F_1(0,y,t),\ F_1(x,y,t)=\begin{cases}\dfrac{1}{\beta_{13}}\dfrac{\partial U_1}{\partial x}-U_1, & 0\le y\le b,\ 0\le x\le\delta\\ -1, & b<y\le 1\end{cases}.$$

In the similar way using the linear combination of the equations (5) together with second BC from (3) we rewrite following BC on the right hand side of the wall:

$$\left(\frac{\partial U_0}{\partial x}+\beta_{21}U_0\right)\Bigg|_{x=\delta-0}=\beta_{21}F_0(\delta,y,t),\quad F_0(x,y,t)=\begin{cases}\left(\frac{1}{\beta_{22}}\frac{\partial U}{\partial x}+U\right), 0\le y\le b, 0\le x\le\delta\\ 0, \qquad\qquad , b<y\le 1.\end{cases},$$

On the assumption that the functions $F_1(0,y,t)$, $F_0(x,y,t)$ are given we can represent solution for the wall in very well known form by the Green function:

$$\begin{aligned}U_0(x,y,t)&=\int_0^\delta\int_0^1 U_0^0(\xi,\eta)G_0(x,y,\xi,\eta,t)d\xi d\eta,\\ &-a_0^2\beta_{11}\int_0^t\int_0^1 F_1(0,\eta,\tau)G_0(x,y,0,\eta,t-\tau)d\eta d\tau+\\ &+a_0^2\beta_{21}\int_0^t\int_0^b F_0(\delta,\eta,\tau)G_0(x,y,\delta,\eta,t-\tau)d\eta d\tau.\end{aligned}\tag{9}$$

Unfortunately the representation (9) is unusable as solution for the wall because of unknown functions $F_1(0,y,t)$, $F_0(x,y,t)$, i.e. temperature in the fins $U(x,y,t)$ and $U_1(x,y,t)$. That is why we will pay attention to the solution for the fins now. In the same way we can rewrite the conjugations conditions in the form of BC on the left side of the right rectangular fin:

$$\left(\frac{\partial U}{\partial x}-\beta_{22}U\right)\Bigg|_{x=\delta+0}=\beta_{22}F(\delta,y,t),\quad F(x,y,t)=\left(\frac{1}{\beta_{21}}\frac{\partial U_0}{\partial x}-U_0\right), 0\le y\le b, \delta\le x\le\delta+l.$$

Then, similar as for the wall we can represent solution for the right fin in following form:

$$\begin{aligned}U(x,y,t)&=\int_\delta^{\delta+l}\int_0^1 U^0(\xi,\eta)G(x,y,\xi,\eta,t)d\xi d\eta-\\ &-a^2\beta_{22}\int_0^t\int_0^b F(\delta,\eta,\tau)G(x,y,\delta,\eta,t-\tau)d\eta d\tau.\end{aligned}\tag{10}$$

Finally, we rewrite the conjugations conditions in the form of BC on the right side of the left rectangular fin:

$$\left(\frac{\partial U_1}{\partial x}+\beta_{13}U_1\right)\Bigg|_{x=0+0}=\beta_{13}F_2(0,y,t),\quad F_2(0,y,t)=\frac{1}{\beta_{11}}\frac{\partial U_0}{\partial x}+U_0,\ 0\le y\le b.$$

So, solution for the left fin we can represent in following form:

$$\begin{aligned}U_1(x,y,t)&=\int_{-l_1}^0\int_0^b U_1^0(\xi,\eta)G_1(x,y,\xi,\eta,t)d\xi d\eta+\\ &a_1^2\beta_{13}\int_0^t\int_0^b G_1(x,y,-l_1,\eta,t-\tau)d\eta d\tau+\\ &+a_1^2\beta_{13}\int_0^t\int_0^b F_2(0,\eta,\tau)G_1(x,y,0,\eta,t-\tau)d\eta d\tau+a_1^2\beta_{13}\int_0^t\int_{-l_1}^0 G_1(x,y,\xi,b,t-\tau)d\eta d\tau.\end{aligned}\tag{11}$$

The expression of the Green functions, eigenfunctions and transcendental equations for in (9), (10), (11) can see in [5].

Using solutions (9), (10), (11) at points $(\delta, y, t), (0, y, t)$, functions $F_1(0, y, t)$, $F_0(x, y, t)$, $F_2(0, y, t)$, $F(x, y, t)$ and some notations we obtain system of Fredholm integral equations of 2nd kind:

$$
\begin{aligned}
&F_0(\delta, y, t) = -a^2 \int_0^t \int_0^b F(\delta, \eta, \tau)\Gamma(\delta, y, \delta, \eta, t-\tau) d\eta d\tau + C_0(y,t), \\
&F_1(0, y, t) = a_1^2 \int_0^t \int_0^b F_2(0, \eta, \tau)\Gamma_1(0, y, 0_1, \eta, t-\tau) d\eta d\tau + C_1(y,t), \\
&F(\delta, y, t) = -a_0^2 \frac{\beta_{11}}{\beta_{21}} \int_0^t \int_0^1 F_1(0, \eta, \tau)\Gamma_0(\delta, y, 0, \eta, t-\tau) d\eta d\tau + \\
&+ a_0^2 \int_0^t \int_0^b F_0(\delta, \eta, \tau)\Gamma_0(\delta, y, \delta, \eta, t-\tau) d\eta d\tau + C(y,t) \\
&F_2(0, y, t) = -a_0^2 \int_0^t \int_0^1 F_1(0, \eta, \tau)\Gamma_2(0, y, 0, \eta, t-\tau) d\eta d\tau + \\
&+ a_0^2 \frac{\beta_{21}}{\beta_{11}} \int_0^t \int_0^b F_0(\delta, \eta, \tau)\Gamma_2(0, y, \delta, \eta, t-\tau) d\eta d\tau + C_2(y,t).
\end{aligned}
\qquad (12)
$$

When solved system of integral equation (12) we immediately can obtain the temperature distributions between fins and wall and then get solutions for (9), (10), (11).

Conclusions

We have constructed several exact three dimensional analytical solutions for a one element of periodical system with rectangular fin where the wall and the fin consist of materials which have different thermal properties. These solutions are in the form of Fredholm integral equation of 2nd kind and has continuous kernel.

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