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About of Blow-up Phenomena for Nonlinear Heat Transfer Problem Between Two Infinite Coaxial Cylinders

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Abstract

We study the numerical methods for solving initial-boundary value problems of some nonlinear heat transfer equations in polar coordinates between two infinite coaxial cylinders (rings) with surfaces $r = r_0,\ r = R > r_0$. Using the method of lines the approximation in the space of corresponding initial boundary value problems in one and two layered domain is based on the finite-difference scheme with central difference (FDS). The original method is used for the calculations of the circular matrices.

We study the behaviour of solutions, which simulate the burning processes, at the time and also when $t \to \infty$ depending of the parameters. In the fixed time moment the solution have "blow up" phenomena – the solution tends to infinity in a small interval or in all domains by a fixed time moment

1. The Mathematical Model

A large number of papers in the time period of 1970 -1990 are devoted to blow-up phenomena in quasilinear parabolic equations [1]. In this paper the 1D initial - boundary problem for nonlinear PDEs in the polar coordinates with radial symmetry of blow-up regimes

$$\frac{\partial u(r,t)}{\partial t} = \frac{\partial}{r\partial r}\left(\lambda r \frac{\partial (u(r,t))^{\sigma+1}}{\partial r}\right) + a(u(r,t))^{\beta},\ r \in [r_0, R],\ t > 0, \tag{1.1}$$

by $\sigma \ge 0, \beta > 0, \lambda > 0, a \ge 0$ and with conditions $u(r_0,t) = u(R,t) = 0,\ u(r,0) = u_0(r) \ge 0$ is considered. We study the behaviour of solutions (1.1) at the time and also when $t \to \infty$, depending on the parameters $\sigma, \beta, \lambda, a$. The corresponding linear transfer problems are considered in [2], [3]. In [4] are solved the equation (1.1) in one layer of Cartesian coordinates.

Let the cylindrical domain $\{(r,\phi,z): r_0 < r < R, 0 \le \phi \le 2\pi, -\infty < z < \infty\}$ contain thermal conducting material, where r_0, R are the radiuses of the coaxial cylinders. The surfaces of these cylinders are with constant temperature $u = 0$. The 2D domain (r,ϕ) with thickness $l = R - r_0$ is multilayer media Ω of $\overline{N}$ layers $\Omega = \{(r,\phi): r \in \Omega_k, k = \overline{1,\overline{N}}, 0 \le \phi \le 2\pi\}$ where each layer is in the form $\Omega_k = \{(r,\phi): r_{k-1} \le r \le r_k, 0 \le \phi \le 2\pi\}, r_{\overline{N}} = R$.

In the 2D case we shall consider the initial - boundary value problem for solving the temperature $u = u(r,\phi,t) \ge 0$ from the following nonlinear heat transfer PDEs:

$$\frac{\partial u}{\partial t} = \lambda \Delta(u^{\sigma+1}) + a u^{\beta},\ r \in (r_0, R), \phi \in [0, 2\pi],\ t > 0, \tag{1.2}$$

where in every layer $\lambda > 0$ is the piece-wise constants coefficient of heat conductivity, $a > 0$ is the constant parameter, Δ is Laplace operator $\Delta * = r^{-1}\partial(r\partial * / \partial r)/\partial r + r^{-2}\partial^2 * / \partial\varphi^2$.

If the initial condition $u(r,\phi,0) = u_0(r)$ depends only on r then the solution $u(r,t)$ is with the radial symmetry (1.1).

In every layer Ω_k the function u and λ are in the form $u_k, \lambda_k, k = \overline{1,\overline{N}}$.

We have the continuity conditions on the interior surfaces $r = r_k$, $r = r_k, k = \overline{1,\overline{N}-1}$: $u_k(r_k,\phi,t) = u_{k+1}(r_k,\phi,t)$, $\lambda_k\partial u_k(r_k,\phi,t)/\partial r = \lambda_{k+1}\partial u_{k+1}(r_k,\phi,t)/\partial r$ and boundary conditions on the exterior surfaces $r = r_0, r = r_{\overline{N}} = R$: $u_1(r_0,\phi,t) = u_{\overline{N}}(R,\phi,t) = 0$. For the initial condition by $t = 0$ we give $u_k(r,\phi,0) = u_{0,k}(r,\phi), k = \overline{1,\overline{N}}$, where $u_{0,k}(r,\phi) \geq 0$ is continuous function in every layer.

2. Some Theoretical Aspects in One and Two Layers by Radial Symmetry

In one layer similarly [1] we consider the following spectral problem $\lambda\partial(r\partial\psi/\partial r)/\partial r + r\mu\psi = 0, \psi(r_0) = \psi(R) = 0$, where μ are the eigenvalues.

The solution of this problem is in the form: $\psi_k(r) = Y_0(b_k r) - \gamma_k J_0(b_k r)$, $\gamma_k = Y_0(b_k r_0)/J_0(b_k r_0), k = 1,2,3,\ldots$, where $b_k = \sqrt{\mu_k/\lambda}$, J_0, Y_0 are the Bessel functions for zero order of the first and the second kind. The eigenvalues μ_k satisfy following transcendental equations: $J_0(b_k r_0)Y_0(b_k R) - J_0(b_k R)Y_0(b_k r_0) = 0$.

From the norm of the first eigenfunction $\psi_1^*(r) = \psi_1(r)/I_0, I_0 = \int_{r_0}^{R} r\psi_1(r)dr$ follows that $\int_{r_0}^{R} r\psi_1^*(r)dr = 1$. Multiplying the equation (1.1) with function $r\psi_1^*(r)$ and integrating it by parts twice we get $dE/dt = a(\psi_1^*, u^\beta) - \mu_1(\psi_1^*, u^{\sigma+1})$, where $(\psi_1^*, f) = \int_{r_0}^{R} r\psi_1^*(r) f(u(r,t))dr$, $E(t) = \int_{r_0}^{R} r\psi_1^*(r)u(r,t)dr \geq 0$, if $u(r,t) \geq 0$, $E_0 = E(0) = \int_{r_0}^{R} r\psi_1^*(r)u_0(r)dr \geq 0$.

Similarly [1] we can proved that by $\beta \geq \sigma + 1$, $\mu_1 E_0^{\sigma+1} < aE_0^\beta$ the solution $u(r,t) \geq 0$ is **unbounded** in the time and exist finite value of T_* when $\max u(r,t) \to \infty$ if $t \to T_*$ (a "blow up" phenomena), where

$$T_* = E_0^{-\sigma}\left(aE_0^{\beta-\sigma-1} - \mu_1\right)^{-1}/(\beta - 1) < \infty. \tag{2.1}$$

If $\beta = \sigma + 1$ and $\mu_1 < a$ then the fixed time moment is

$$T_* = (\sigma(a - \mu_1))^{-1} E(0)^{-\sigma} \tag{2.2}$$

If $\beta < \sigma + 1$ then the solution is bounded for finite value of $t < \infty$.

Similarly for **two layers** ($\overline{N} = 2, r_1 = H, r_2 = R$), we consider the following spectral problem $\lambda\partial(r\partial\psi/\partial r)/\partial r + r\mu\psi = 0, \psi^1(r_0) = \psi^2(R) = 0$, where $\psi(r) = \{\psi^1(r)\ (r_0 \leq r \leq H); \psi^2(r)\ (H \leq r \leq R)\}, \lambda = \{\lambda_1; \lambda_2\}, \psi^1(H) = \psi^2(H)$, $\lambda_1\partial\psi^1(H)/\partial r = \lambda_2\partial\psi^2(H)/\partial r$. The solution of this problem is in the form: $\psi_k^1(r) = \gamma_k\left(Y_0(b_k^1 r_0)J_0(b_k^1 r) - Y_0(b_k^1 r)J_0(b_k^1 r_0)\right)$, $\psi_k^2(r) = Y_0(b_k^2 R)J_0(b_k^2 r) - Y_0(b_k^2 r)J_0(b_k^2 R)$,

$k=1,2,3,...$, where $b_k^1=\sqrt{\mu_k/\lambda_1}$, $b_k^2=\sqrt{\mu_k/\lambda_2}$, $\gamma_k=\dfrac{Y_0(b_k^2R)J_0(b_k^2H)-Y_0(b_k^2R)J_0(b_k^2R)}{Y_0(b_k^1r_0)J_0(b_k^1H)-Y_0(b_k^1H)J_0(b_k^1r_0)}$.

The eigenvalues μ_k satisfy following transcendental equations

$$\begin{aligned}&\lambda_1 b_k^1(Y_0(b_k^2R)J_0(b_k^2H)-Y_0(b_k^2H)J_0(b_k^2R))(Y_0(b_k^1r_0)J_1(b_k^1H)-Y_1(b_k^1H)J_0(b_k^1r_0))-\\&-\lambda_2 b_k^2(Y_0(b_k^1r_0)J_0(b_k^1H)-Y_0(b_k^1H)J_0(b_k^1r_0))(Y_0(b_k^2R)J_1(b_k^2H)-Y_1(b_k^2H)J_0(b_k^2R)=0\end{aligned} \quad (2.3)$$

where J_1, Y_1 are the Bessel functions for first order of the first and the second kind. From the norm of the first eigenfunction $\psi_1^*(r)=\psi_1(r)/I_0$, $\psi_1^*=\{\psi_1^{1*};\psi_1^{2*}\}$, $\psi_1=\{\psi_1^1;\psi_1^2\}$,

$I_0=\int_{r_0}^R r\psi_1(r)dr=\int_{r_0}^H r\psi_1^1(r)dr+\int_H^R r\psi_1^2(r)dr$,

follows that $\int_{r_0}^R r\psi_1^*(r)dr=\int_{r_0}^H r\psi_1^{1*}(r)dr+\int_H^R r\psi_1^{2*}(r)dr=1$.

Multiplying the equation (1.1) by function $r\psi_1^*(r)$ and integrating it by parts twice both integrals we obtain the expressions (2.1, 2.2).

3. Methods of Lines and FDS for the One and Two Layers

For numerical calculation in the one layered domain $(\lambda_k=\lambda, u_k=u)$ we consider uniform grid with additional grid points $(r_k=kh+r_0, k=\overline{0,N}, Nh=R-r_0)$. We consider two cases: the 1D problem with radial symmetry and the 2D problem in the space. For solving the equation (1.2) with radial symmetry we use the method of lines to reduce the nonlinear heat transfer problem to initial value problem for system of nonlinear ODEs of the first order. For the 2D problem we obtain the stationary solutions using the vector finite difference scheme with circulant matrix.

In the 1D case from (1.2) we can be directly obtain the system of nonlinear ODEs with the second order of approximation in the space in the following matrix form

$$\dot{U}=(\lambda/h^2)AG+F, \quad (3.1)$$

where A is the standard 3-diagonal matrix of $N-1$ order with the non zero elements $a_{k,k}=-2$, $a_{k,k+1}=(r_{k+0.5})/r_k$, $a_{k-1,k}=(r_{k-0.5})/r_k$, $G,F,\dot{U}$ are the column-vectors of $N-1$ order with elements $g_k=(u(x_k,t))^{\sigma+1}$, $f_k=a(u(x_k,t))^{\beta}$, $\dot{u}_k=\dot{u}(x_k,t)$.

In the 2D case using the transformation $V(t,r,\phi)=u^{\sigma+1}(t,r,\phi)$ and the method of stationarity in the equation (1.2) we approximate the derivative $\partial V/\partial t$ by the difference $(V_{i+1}(r,\phi)-V_i(r,\phi))/\tau$, where $i=0,1,...,I$, τ is the parameter of iterations.

The number of iterations I is determined from following conditions: $|V_{I+1}(r,\phi)-V_I(r,\phi)|\le\varepsilon$, where ε is the desirable precision. We can rewrite for the each iteration the heat transfer problem in the following form

$$\begin{cases}(V_{i+1}(r,\phi)-V_i(r,\phi))/\tau=\lambda\Delta V_{i+1}(r,\phi)+a(V_i(r,\phi))^{\alpha}, i=0,1,...,I,\\ V_0(r,\phi)=u_0(r,\phi), V_i(r_0,\phi)=V_i(R,\phi)=0, V_i(r,\phi+2\pi)=V_i(R,\phi)\end{cases}, \quad (3.2)$$

where $\alpha=\beta/(\sigma+1)$, the function $u_0(r,\phi)=V_0(r,\phi)$ is the initial condition for the iterations.

We consider an uniform grid: $\omega_h=\{(r_k,\phi_j), r_k=r_0+kh, \phi_j=jh_\phi\}$, $k=\overline{0,N}$, $j=\overline{1,M}$, $r_0+Nh=R$, $Mh_\phi=2\pi$. The equation (3.2) in the grid points (r_k,ϕ_j) is replaced by vector difference equations of second order approximation in 5-point stencil:

$$A_k V_{i+1,k-1} - C_k V_{i+1,k} + B_k V_{i+1,k+1} + F_{i,k} = 0,\ V_{i+1,0} = V_{i+1,N} = 0, \tag{3.3}$$

where $V_{i,k}, F_{i,k}$ are column-vectors with components $v_{i,k,j} \approx V_i(r_k, \phi_j)$, $f_{i,k,j} = a(v_{i,k,j})^\alpha + v_{i,k,j}/\tau,\ k = \overline{1, N-1},\ j = \overline{1, M}$, A_k, B_k, C_k are the circulant symmetrical matrices with M-order: $A_k = [a_{k,1}, 0, 0, ..., 0]$, $B_k = [b_{k,1}, 0, 0, ..., 0]$, $C_k = [c_{k,1}, c_{k,2}, 0, 0, ..., 0, c_{k,2}]$, where $a_{k,1} = r_{k-0.5}/r_k h^2$, $b_{k,1} = r_{k+0.5}/r_k h^2$, $c_{k,2} = c_{k,M} = -(r_k^2 h_\phi^2)^{-1}$, $c_{k,1} = a_{k,1} + b_{k,1} - 2c_{k,2} + \tau^{-1}$.

Using special arithmetical operations with circulant matrices the finite vector difference scheme (3.3) is solved by the Gauss elimination method.
Similarly we consider the **two layered domain** $(\Omega_1, \Omega_2, \lambda_1 \neq \lambda_2)$ and uniform grid in every layer with grid points $\left(r_k = kh + r_0, k = \overline{0, N}, Nh = R - r_0, Kh + r_0 = H = r_1 < R = r_2\right)$.

In the 1D case we obtain the system of nonlinear ODEs in the following matrix form (3.1), where A is the block matrix of $N-1$ order with two blocks of 3-diagonal matrix form of $K-1$ and $N-K$ orders.

In the 2D case similarly (3.2) follows the heat transfer problem in iterations form

$$\begin{cases} \left(V_{i+1}^m(r,\phi) - V_i^m(r,\phi)\right)/\tau = \lambda_m \Delta V_{i+1}^m(r,\phi) + a(V_i^m(r,\phi))^\alpha,\ i = 0,1,...,I \\ V_0^m(r,\phi) = u_0(r,\phi), V_i^1(r_0,\phi) = V_i^2(R,\phi) = 0, V_i^m(r,\phi+2\pi) = V_i^m(R,\phi), \\ V^1(H,\phi) = V^2(H,\phi), \lambda_1 \partial V^1(H,\phi)/\partial r = \lambda_2 \partial V^2(H,\phi)/\partial r \end{cases} \tag{3.4}$$

where V^1, V^2 are corresponding the solutions in the domains $\Omega_1, \Omega_2, m = 1,2$. The heat transfer equation (3.4) in the uniform grid (r_k, ϕ_j) can be rewritten in the matrix-vector form (3.3).

4. Some Examples and Numerical Results

The numerical experiment for the linear equation (1.2) with $\sigma = 0$, $f = \sin(\phi), a = 3$, $\lambda_1 = 1; 100$, $\lambda_2 = 1$ and $u_0(r,\phi) = (R-r)(r-r_0) \geq 0$, $R = 1$, $r_0 = 0.2$, $H = 0.6$ is compared with the following stationary analytical solution $u_1(r,\phi) = C_1 r + C_2 r^{-1} - r^2/\lambda_1$, $u_2(r,\phi) = C_3 r + C_4 r^{-1} - r^2/\lambda_2$, where C_1, C_2, C_3, C_4 are constants.

For the radial symmetry case is used also nonlinear test with $\beta = 0$. The stationary solution is in the form: $u_1(r) = \left(C_1 \ln r + C_2 - 0.25ar^2/\lambda_1\right)^\alpha$, $u_2(r) = \left(C_3 \ln r + C_4 - 0.25ar^2/\lambda_1\right)^\alpha$, where $\alpha = (\sigma+1)^{-1}$. The numerical results are agreed with 4 decimal signs with respect to analytical solutions. From the numerical results follows that the minimal value of error is by $N = M$ and further the calculations are produced by different value of σ, β and $N = M = 80, \varepsilon = 10^{-4}$.
We can obtain the four type solutions (radial symmetry) depending on the parameters α, β, a and with $\lambda_1 = 100$, $\lambda_2 = 1$, $\mu_1 = 59.2001$, $\lambda_1 = 1$, $\lambda_2 = 100$, $\mu_1 = 58.9950$ (in two layers):

1) $\sigma = 3, \beta = 5, a = \mu_1$ the stationary solution $u_{st}(r)$ is zero,

2) $\sigma = 3, \beta = 4, a = \mu_1$, the solution $u_{st}(r) \neq 0$ if $t \to T_{st} < \infty$,

3) $\sigma = 3, \beta = 4, a = 60, a > \mu_1, u(r,t) \to \infty$ globally for all r when $t \to T_* < \infty$ for $T_* = 268.9988$ (theoretical value $T_{**} = 293.8056$).

4) $\sigma = 3, \beta = 5, a = 500$, $u(r,t) \to \infty$ locally, when $t \to T_* < \infty$ for $T_* = 32.44096$ of point $r = 0.75$ if $\lambda_1 = 100$, $\lambda_2 = 1$ and for $T_* = 14.46177$ of point $r = 0.25$ if $\lambda_1 = 1, \lambda_2 = 100$ (Fig. 1., 2.).

Tab. 1. The values of T_* (numerical value) and T_{**} (theoretical value) by $a = 120$

$\lambda_1 = \lambda_2 = 1$					$\lambda_1 = 100, \lambda_2 = 1$		
σ	β	T_*	T_{**}	Q	T_*	T_{**}	Q
1	2	0.06822	0.0729	8.2405	0.13804	0.1464	2.0270
1	3	0.70117	306554	1.0722	$u_{st} = 0$	$u_{st} = 0$	0.2277
2	3	0.23933	0.2801	8.2405	0.57820	0.6514	2.0270
3	4	1.09944	1.4354	8.2405	3.1775	3.8654	2.0270
4	5	5.61870	8.2744	8.2405	19.3153	25.8023	2.0270

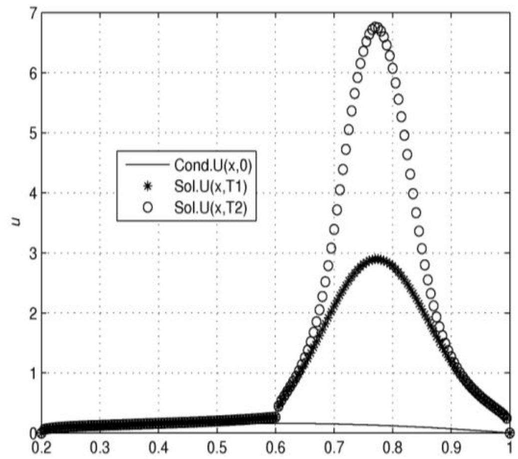

Fig. 1. Solution $u \to \infty$ for $r = 0.75, \beta = 5, \sigma = 3, a = 500, \lambda_1 = 100, \lambda_2 = 1$

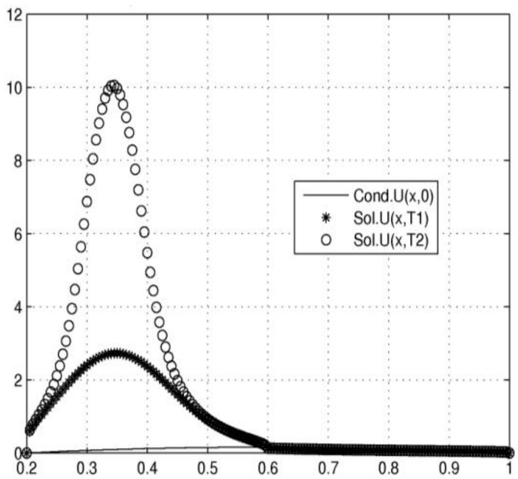

Fig. 2. Solution $u \to \infty$ for $r = 0.25, \beta = 5, \sigma = 3, a = 500, \lambda_1 = 1, \lambda_2 = 100$

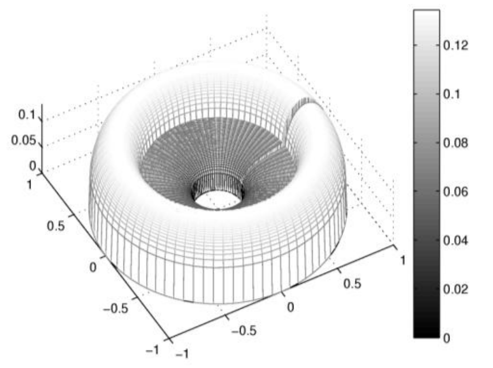

Fig. 3. 2D solution $u_{st} \neq 0$ $\beta = 4$, $\sigma = 3$, $a = \mu_1 = 59.2001$, $\lambda_1 = 100$, $\lambda_2 = 1$

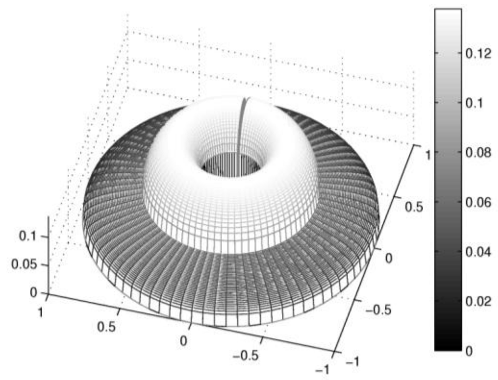

Fig. 4. 2D solution $u_{st} \neq 0$ $\beta = 4$, $\sigma = 3$, $a = \mu_1 = 58.9950$, $\lambda_1 = 1$, $\lambda_2 = 100$

If $\beta < \sigma + 1$**,** then for all $a > 0$ we have by $t \to T_{st} < \infty$ the stationary solution $u_{st}(r)$ **If** $\beta = \sigma + 1$, then for all $a < \mu_1, u_{st}(r) = 0$. If $a = \mu_1$, then the convergence to stationary solution is very fast in the time. If $a > \mu_1$, then the solutions is unbounded in the time $t \geq T_*$ in all interval $r \in (r_0, R)$ (T_* is finite value, see the theoretical estimation (2.2)).

If $\beta > \sigma + 1$, then we have "blow up" phenomena by sufficient large value of $E(0)$ or a, when the solution tends to infinity locally in small neighbourhood of interior point in segment $[r_0, R]$. From the theoretical estimation $Q = aE(0)^{\beta-\sigma-1}/\mu_1 > 1$ follows that the solution is unbounded in the time $t \geq T_* < \infty$, where $T_* = E_0^{-\sigma}(aE_0^{\beta-\alpha-1} - \mu_1)^{-1}/(\beta - 1)$. The behaviour of the solution for $\sigma + 1 \leq \beta$ we can see in the table 1 (T_* is numerical value, T_{**} is theoretical value).

For the 2D case $(\beta = 4, \sigma = 3, a = \mu_1)$ the stationary solution is independent on the azimuthal coordinate ϕ (Fig. 3. 4.). These pictures are obtained by τ=0.01; 0.001; 0.0005 and I=20; 40; 70.

5. Conclusions

The nonlinear heat transfer problem is approximated with the nonlinear initial value problems of a system of ODEs of the first order. Depending on the parameters two types of solutions are obtained:

1) for large value of the time t the solution is stationary or tends to zero;

2) in the fixed time moment the solution have blow up phenomena – the solution tends to infinity in a small interval or in all domain by a fixed time moment.

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