International Scientific Colloquium
Modelling for Material Processing
Riga, June 8-9, 2006

Calculations of the Thermal-conductivity Coefficients for 1-D Heat Transfer Inverse Problems

H. Kalis, Sh. Guseinov

Abstract

As a rule, coefficient inverse problems, in general case, are ill-posed problems, and in order to determine the required unknown coefficients, the conforming additional conditions are necessary. In present work we consider a some class of homogeneous one-dimensional on spatial variable coefficient inverse problems of thermal conductivity in the bounded domain under some additional information. Authors offer one simple numerical method for determination of required coefficient of thermal diffusivity. Let us note the offered method is suitable for solution of some classes of nonhomogeneous coefficient inverse problems and also nonlinear direct and inverse problems of mathematical physics.

Introduction

In order to determine the thermal diffusivity in a homogeneous medium, with other characteristics of the process of heat propagation being known, the so-called "Transient Hot-Strip Method" (or THS method, see e.g. [1-3] and references therein) can be used, which is applicable for solids and fluids with low electric conductivity. According to the method, between two halves of the specimen, whose thermal characteristics should be found, a thin metallic strip is clamped. Such a strip is used both as continuous plane resistive source of heat and as the sensor of temperature increase in the strip itself. In the mentioned works [1-3] a homogeneous material is considered. In work [3] the problem is solved by a numerical method – namely, by the method of finite elements. In [4] and [5], similar to the present work, a two-layer medium is analyzed. As opposed to [3], in the works [4-5] we apply an analytical method of solution, thus narrowing the problem – namely, we determine only two coefficients of thermal conductivity assuming the thermal capacity to be known. In [6] principle another approach for the solution of considered inverse problems is proposed. In [7] the proposed new approach was applied (without any numerical calculations) to different direct or inverse linear or nonlinear problems of mathematical physics. In this paper we will show an adaptability of this method with the help of numerical calculations.

1. Mathematical Statement of Direct and Corresponding Inverse Problems

Let consider the following one-dimensional boundary-value problem:

$$\frac{\partial u}{\partial t} = a^2 \cdot \frac{\partial^2 u}{\partial x^2} + f(x,t) \ , \ 0 < x < L < \infty \ , \ 0 < t \le T \ , \tag{1.1}$$

$$u(x,t)\big|_{t=0} = h(x) \ , \ 0 \le x \le L \ , \tag{1.2}$$

$$u(x,t)\big|_{x=0} = \gamma_1(t) \ , \ t \in \left[0,T\right] , \tag{1.3}$$

$$u(x,t)\Big|_{x=L} = \gamma_2(t),\ t \in [0,T]. \tag{1.4}$$

In the formulated problem (1.1)-(1.4) we suppose that the functions $h(x), \gamma_1(t), \gamma_1(t), f(x,t)$ are known, and they satisfy the conditions $\gamma_1(0) = h(0) = 0$, $\gamma_2(0) = h(L)$, $h(x) \in C^1[0,L]$,

$\gamma_1(t), \gamma_2(t) \in C^2[0,T]$, $f(x,t) \in C[0,L]\times[0,T]$. (*)

Then the direct problem (1.1)-(1.4) will be uniquely determined by numeric parameter a^2.

Now let us formulate corresponding inverse problem: it is necessary to find a thermal diffusivity a^2 and a function $u(x,t)$, $(x,t) \in D \overset{def}{\equiv} [0,L]\times[0,T]$ from (1.1)-(1.4) under the conditions (*) and $u(x,t),\ u_t(x,t),\ u_x(x,t),\ u_{xx}(x,t) \in C\{D\}$. Moreover we assume that the following additional information have a place:

$$\frac{\partial u(x,t)}{\partial x}\bigg|_{x=0} = 0,\ t \in [0,T]. \tag{1.5}$$

2. Analytic method for the Solution of Inverse Problem

To solve the formulated inverse problem (1.1)-(1.5) we will represent unknown function $u(x,t)$ in the following form:

$$u(x,t) = U^{(1)}(t) + x\cdot U^{(2)}(t) + \int_0^x (x-\xi)\cdot U^{(3)}(\xi,t)d\xi, \tag{2.1}$$

where $U^{(1)}(t)$, $U^{(2)}(t)$ and $U^{(3)}(x,t) \neq 0$ are not for a while yet known functions. After substituting boundary condition (1.3) and additional condition (1.5) in the representation (2.1) we get

$$\gamma_1(t) - \gamma_1(0) + \int_0^x (x-\xi)\cdot\left[U^{(3)}(\xi,t) - U^{(3)}(\xi,0)\right]d\xi = a^2\cdot\int_0^t U^{(3)}(x,\tau)d\tau + \int_0^t f(x,\tau)d\tau.$$

From here with the help of initial condition (1.2) we have

$$\int_0^T dt\int_0^L \gamma_1(t)dx + \int_0^T dt\int_0^L dx\int_0^x (x-\xi)\cdot U^{(3)}(\xi,t)d\xi - \int_0^T dt\int_0^L h(x)dx = a^2\cdot\int_0^T dt\int_0^L dx\int_0^t U^{(3)}(x,\tau)d\tau +$$
$$+\int_0^T dt\int_0^L dx\int_0^t f(x,\tau)d\tau.$$

After not complicated transformation we can write that

$$L\cdot\int_0^T \gamma_1(t)dt + \frac{1}{2}\cdot\int_0^L (L-x)^2dx\int_0^T U^{(3)}(x,t)dt - T\cdot\int_0^L h(x)dx - \int_0^L dx\int_0^T (T-t)\cdot f(x,t)dt =$$
$$= a^2\cdot\int_0^L dx\int_0^T (T-t)\cdot U^{(3)}(x,t)dt.$$

From here

$$a^2 = \frac{L\cdot\int_0^T \gamma_1(t)dt + \frac{1}{2}\cdot\int_0^L (L-x)^2dx\int_0^T U^{(3)}(x,t)dt - T\cdot\int_0^L h(x)dx - \int_0^L dx\int_0^T (T-t)\cdot f(x,t)dt}{\int_0^L dx\int_0^T (T-t)\cdot U^{(3)}(x,t)dt}. \tag{2.2}$$

From formulae (2.2) we can see that an unknown diffusivity coefficient a^2 is uniquely determined if a function $U^{(3)}(x,t)$ is known. An unknown function will be determined from the first kind Fredholm integral equation

$$\int_0^L (L-x)\cdot U^{(3)}(x,t)dx = \gamma_2(t) - \gamma_1(t)\,. \tag{2.3}$$

After designation

$$\gamma(t) \overset{def}{\equiv} \gamma_2(t) - \gamma_1(t)\,,$$

$$A\bullet \overset{def}{\equiv} \int_0^L (L-x)\bullet dx\,,$$

the equation (2.3) will be rewritten in the form of operator equation

$$\left(AU^{(3)}\right)(t) = \gamma(t)\,. \tag{2.4}$$

An operator equation belong to a class of the first kind operator equation. Consequently this equation is an ill-posed problem. Having applied to (2.4) one of regularizing method, for example, the Tikhonov's regularization method (see [7-8]), we can find the regularized solution $U_{REGUL.}(x,t)$ (more precisely, the normal solution) of this problem. Further, considering $U_{REGUL.}(x,t)$ in (2.3), we will easily find unknown coefficient of thermal diffusivity a^2, and then according to formula (2.1) we can also determine an unknown function $u(x,t)$.

3. The Solution of Inverse Problems in the Presence of Other Additional Information

Now we will consider the following inverse problem: it is necessary to determine an unknown diffusivity coefficient a^2 and also an unknown function $u(x,t),\ (x,t) \in D \overset{def}{\equiv} [0,L]\times[0,T]$ from

$$\frac{\partial u}{\partial t} = a^2 \cdot \frac{\partial^2 u}{\partial x^2} + f(x,t)\,,\ 0 < x < L\,,\ 0 < t \le T\,, \tag{3.1}$$

$$u(x,t)\big|_{t=0} = h(x)\,,\ 0 \le x \le L\,, \tag{3.2}$$

$$u(x,t)\big|_{x=0} = \gamma_1(t)\,,\ t \in [0,T]\,, \tag{3.3}$$

$$u(x,t)\big|_{x=L} = \gamma_2(t)\,,\ t \in [0,T] \tag{3.4}$$

under the additional conditions

$$\Theta[x_0]u(x,t) = \gamma_3(t) \in C^k[0,T]\ \left(k \in \{1,2\}\right), \tag{3.5}$$

where $\Theta[x_0]$ is a linear operator that acts at the fixed point $x_0 \in [0,L]$, for example, a differential operator on variable x (in the case of $k=2$) or an integral operator with limit of integration from zero to fixed point $x_0 \in [0,L]$ (in the case of $k=1$).

Here we must assume that $u(x,t),\ u_t(x,t),\ u_x(x,t),\ u_{xx}(x,t) \in C\{[0,L]\times[0,T]\}$ and the functions $h(x), \gamma_1(t), \gamma_1(t), f(x,t)$ are given functions: $\gamma_1(0) = h(0)$, $\gamma_2(0) = h(L)$, $h(x) \in C^1[0,L]$, $\gamma_1(t), \gamma_2(t) \in C^2[0,T]$, $f(x,t) \in C[0,L]\times[0,T]$.

Similarly to previous section, we can determine an unknown diffusivity coefficient a^2 by formulae

$$a^2 = \frac{L\cdot\int_0^T[\gamma_2(t)+\gamma_1(t)]dt + L\cdot\int_0^L x\cdot(x-L)dx\int_0^T U^{(3)}(x,t)dt - 2\cdot T\cdot\int_0^L h(x)dx - 2\cdot\int_0^L dx\int_0^T (T-t)\cdot f(x,t)dt}{2\cdot\int_0^L dx\int_0^T (T-t)\cdot U^{(3)}(x,t)dt}, \quad (3.6)$$

where an unknown function $U^{(3)}(x,t)$ is the regularized solution of the following operator equation:

$$\Theta\left[\int_0^{x_0}(x_0-\xi)\cdot U^{(3)}(\xi,t)d\xi\right] - \Theta\left[\frac{x_0}{L}\cdot\int_0^L (L-\xi)\cdot U^{(3)}(\xi,t)d\xi\right] = \gamma_3(t) -$$

$$-\Theta[\gamma_1(t)] - \Theta\left[\frac{x_0}{L}\cdot(\gamma_2(t)-\gamma_1(t))\right]. \quad (3.7)$$

Over again let us consider tke inverse problem (3.1)-(3.5). If instead of the condition (3.3) we have the second type boundary condition

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \gamma_1(t) \in C^1[0,T],$$

and instead of (3.4) there is the second type condition

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=L} = 0\ ,\ t\in[0,T],$$

and finally instead the additional information (3.5) we have additional condition

$$\Theta[x_0]u(x,t) = \gamma_3(t) \in C^k[0,T] \quad (k\in\{1,2\}),$$

where an operator has the same meaning as in the case of an operator in (3.5), then an unknown diffusivity coefficient a^2 can be finding by formulae

$$a^2 = \frac{2\cdot L\cdot\int_0^T U^{(1)}(t)dt - 2\cdot T\cdot\int_0^L h(x)dx + L^2\cdot\int_0^T \gamma_1(t)dt}{2\cdot\int_0^L dx\int_0^T (T-t)\cdot U^{(3)}{}_{REGUL.}(x,t)dt} +$$

$$+\frac{\int_0^L (L-x)^2 dx\int_0^T U^{(3)}{}_{REGUL.}(x,t)dt - 2\cdot\int_0^L dx\int_0^T (T-t)\cdot f(x,t)dt}{2\cdot\int_0^L dx\int_0^T (T-t)\cdot U^{(3)}{}_{REGUL.}(x,t)dt},$$

where the function $U^{(3)}{}_{REGUL.}(x,t)\neq 0$ is the regularized solution corresponding operator equation.

4. Numerical Calculations by the Example of the Solution of Inverse Problem

As a case in point, we will consider the coefficient inverse problem (3.1)-(3.5), where (3.5) is the integral condition

$$\int_0^L u(x,t)dx = \gamma_3(t). \quad (3.8)$$

Then from (3.7) following the first kind Fredholm integral equation for determination of an unknown function $U^{(3)}(x,t)$:

$$\gamma_3(t)=\frac{L}{2}\cdot(\gamma_1(t)+\gamma_2(t))-\frac{1}{2}\cdot\int_0^L x\cdot(L-x)\cdot U^{(3)}(x,t)dx. \qquad (3.9)$$

Now we assume that

$$f(x,t)=0,\ \gamma_1(t)=\gamma_2(t)=0,\ \gamma_3(t)=e^{-kt}\ (k>0),\ h(x)=\sin(\pi x),\ L=T=1.$$

The solution of our direct problem is the function $u_{ex.}(x,t)=e^{-a^2\pi^2 t}\cdot\sin(\pi x)$. From (3.9) we get the normal solution of this integral equation is $U^{(3)}(x,t)=-12\cdot e^{-kt}$. After integrating (3.9) on variable t from zero to 1 we can write that

$$\int_0^1\int_0^1 x\cdot(1-x)\cdot U^{(3)}(x,t)dxdt=-\frac{2}{k}\cdot\left(1-e^{-k}\right).$$

Similarly we have get

$$\int_0^1\int_0^1 (1-t)\cdot U^{(3)}(x,t)dxdt=-\frac{12}{k}\cdot\left(1-\frac{1}{k}\left(1-e^{-k}\right)\right) \text{ and } \int_0^1 h(x)dx=\frac{2}{\pi}.$$

Now substitute received results in the expression (3.6) we get

$$a_{reg}^2=\frac{\frac{2}{\pi}-\frac{1}{k}\left(1-e^{-k}\right)}{\frac{12}{k}\cdot\left(1-\frac{1}{k}\left(1-e^{-k}\right)\right)}. \qquad (3.10)$$

Now from (2.1) we have get the approximation solution in the form $u_{reg}(x,t)=6\cdot x\cdot(1-x)\cdot e^{-kt}$. Averaging on the space variable x for the function $u_{ex}(x,t)$ givs us the following formula:

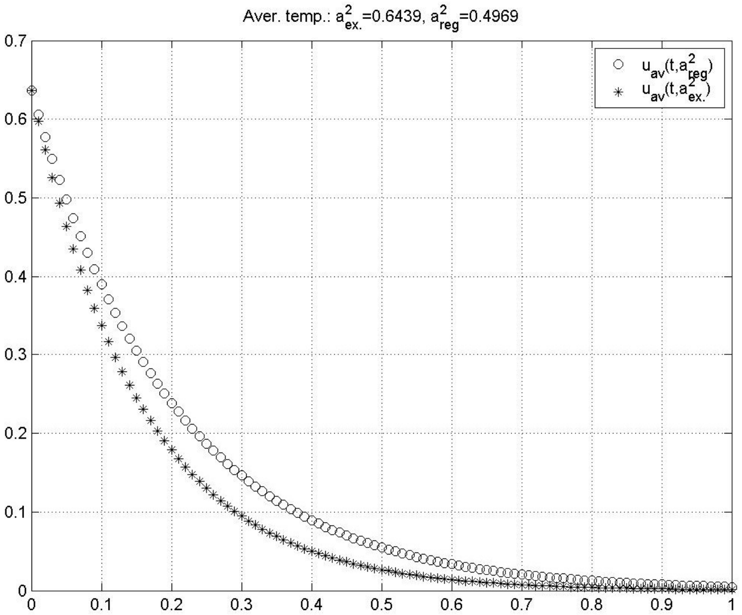
$$u_{av}(t,a)=\int_0^1 u_{ex.}(x,t)dx=\frac{2}{\pi}\cdot e^{-a^2\pi^2 t}. \qquad (3.11)$$

If we substitute in (3.8) the exact solution $u_{ex.}(x,t)$ we have that $\frac{2}{\pi}\cdot e^{-a_{ex.}^2\pi^2 t}=e^{-kt}$. From here after integrating on variable t from zero to 1 we get the following transcendental equation for determination of $a_{ex.}^2$: $\frac{2}{\pi}\cdot\frac{1-e^{-a_{ex.}^2\pi^2}}{a_{ex.}^2\pi^2}=\frac{1-e^{-k}}{k}$. (3.12)

From (3.10) and (3.12) under each fixed values of $k\in\{9,10,11,12\}$ we get the following table for values of $a_{ex.}^2$ and a_{reg}^2:

k	**9**	**10**	**11**	**12**
$a_{ex.}^2$	0.4434	0.4969	0.5503	0.6036
a_{reg}^2	0.5787	0.6439	0.7089	0.7737

With the help of MATLAB under $k=10$ we have the following figure for the functions $u_{av}(t,a_{ex.}^2)$, $u_{av}(t,a_{reg}^2)$:



Remark. If we consider a nonhomogeneous equation (1.1) (or (1.3)) under $f(x,t) = 2 \cdot C \cdot a^2$, $h(x) = \sin(\pi x) + Cx(1-x)$, where C is any constant, then we have $u_{av}(t,a) = \dfrac{C}{6} + \dfrac{2}{\pi} \cdot e^{-a^2\pi^2 t}$, moreover, the expressions (3.10) and (3.12) are no changed (i.e. they are valid also this case).

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Authors

Dr.h.-Math., Dr.h.-Phys. Kalis, Harijs
Dr.-Math. Guseinov, Sharif
Faculty of Physics and Mathematics
University of Latvia
Zellu str. 8
LV-1002 Riga, Latvia
E-mail: kalis@lanet.lv, sh.e.guseinov@inbox.lv