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Modelling of Intensive Steel Quenching Process by Time Inverse Hyperbolic Heat Conduction

A. Buikis, Sh. Guseinov, M. Buike

Abstract

In this paper the hyperbolic heat equation for finite cylinder in two-dimensional approximation is studied. The second initial condition: the speed of the temperature change at the initial moment is not known. Instead of this condition there is given the temperature distribution at the final moment of time. At the beginning the problem of the determination of the temperature change at the initial moment is reduced to the Fredholm integral equation of first kind concerning the second time derivative. The kernel of this integral equation is the Green function for the corresponding classical heat equation. At the end, the solution is written in the closed form.

Introduction

Some quick industrial processes like laser impulse affecting the metal surface are mathematically modelled by hyperbolic heat equation. The process of intensive steel quenching [1]-[3] is another process where the temperature is changing very quickly. In our paper [4] it is described by hyperbolic heat conduction equation in one-dimensional case. Here the process for finite cylinder in two-dimensional approximation is modelled. Because of practical difficulties to learn the speed of the temperature change at initial moment, the approach how to find this unknown function is proposed here.

1. The Mathematical Model of the Intensive Steel Quenching

The model for intensive steel quenching according to the steel temperature $u(r,z,t)$ is mathematically described as follows:

$$\tau_r \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = a^2 \left[r^{-1} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} \right] + f(r,z,t),\ r \in (0,R),\ z \in (0,H),\ t \in (0,T] \quad (1.1)$$

$$\left(\frac{\partial u}{\partial r} + k_1 u \right) = \gamma_1(z,t),\ r = R, z \in [0,H], t \in [0,T], \quad (1.2)$$

$$\frac{\partial u}{\partial z} + k_2 u = \gamma_2(r,t),\ z = H, r \in [0,R], t \in [0,T], \quad (1.3)$$

$$\frac{\partial u}{\partial z} - k_3 u = -\gamma_3(r,t), z = 0, r \in [0,R], t \in [0,T], \quad (1.4)$$

$$\frac{\partial u}{\partial r} \to 0, r \to 0, z \in [0, Z], t \in [0, T], \tag{1.5}$$

$$u = u^0(r, z), t = 0, r \in [0, R], z \in [0, H]. \tag{1.6}$$

The second needed initial condition:

$$\frac{\partial u}{\partial t} = V_0(r, z), t = 0, r \in [0, R], z \in [0, H] \tag{1.7}$$

is unknown and must be found. Instead of (1.7) the temperature distribution at the end of the process of the intensive steel quenching is given there – the following condition:

$$u(r, z, T) = U(r, z), r \in [0, R], z \in [0, H]. \tag{1.8}$$

It formally means that we have Dirichlet "boundary" conditions (1.6), (1.8) according to the time variable. In reality the parameter τ_r is very small in comparison with time scale and the durability of quenching process:

$$\tau_r << \min(1, T).$$

This signifies that we practically have to solve the time inverse problem for classical heat equation after very short time interval.

2. The Solution of the Problem

We rewrite the main equation (1.1) in the form of the heat equation with unknown source term:

$$\frac{\partial u}{\partial t} = a^2 \left[r^{-1} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} \right] + F(r, z, t, \tau_r, u). \tag{2.1}$$

Here $F(r, z, t, \tau_r, u) = f(r, z, t) - \tau_r W(r, z, t),\ W(r, z, t) = \frac{\partial^2 u}{\partial t^2}$.

The solution of the problem (2.1), (1.2)-(1.6) with known source function F for the classical heat conduction equation has the well known form by employing the Green function $G(r, \rho, z, \varsigma, t)$ for the finite cylinder with mixed type (second and third type) boundary conditions:

$$G(r, \rho, z, \varsigma, t) = G_1(r, \rho, t) \cdot G_2(z, \varsigma, t).$$

Here

$$G_1(r, \rho, t) = -\frac{\pi}{4} \cdot \sum_{n=1}^{\infty} \mu_n^2 \cdot e^{-\mu_n^2 \cdot a^2 \cdot t} \cdot J_0(\mu_n \cdot r) J_0(\mu_n \cdot \rho),$$

$$G_1(z, \varsigma, t) = \sum_{m=1}^{\infty} \frac{\left[\cos(\lambda_m \cdot z) + \frac{k_3}{\lambda_m} \cdot \sin(\lambda_m \cdot z) \right] \cdot \left[\cos(\lambda_m \cdot \varsigma) + \frac{k_3}{\lambda_m} \cdot \sin(\lambda_m \cdot \varsigma) \right]}{\frac{k_2}{2 \cdot \lambda_m^2} \cdot \frac{\lambda_m^2 + k_3^2}{\lambda_m^2 + k_2^2} + \frac{k_3}{2 \cdot \lambda_m^2} + \frac{H}{2} \cdot \left(1 + \frac{k_3^2}{\lambda_m^2} \right)} \cdot e^{-a^2 \cdot \lambda_m^2 \cdot t},$$

where $\mu_n > 0$ and $\lambda_m > 0$ are roots of following transcendental equations:

$$\mu \cdot J_1(\mu \cdot R) - k_1 \cdot J_0(\mu \cdot R) = 0,$$

$$\left(\lambda - \frac{k_2 \cdot k_3}{\lambda}\right) \cdot tg(\lambda \cdot H) = k_2 + k_3 .$$

This fact allows us to write the solution in short form as the expression:

$$u(r,z,t) = \Gamma(r,z,t) - 2 \cdot \pi \cdot \tau_r \cdot \int_0^t d\tau \int_0^H d\varsigma \int_0^R \rho G(r,\rho,z,\varsigma,t-\tau) \cdot W(\rho,\varsigma,\tau) d\rho. \qquad (2.2)$$

Here the function $\Gamma(r,z,t)$ contains all the other integrals where Green function is multiplied by known boundary and initial conditions:

$$\Gamma(r,z,t) = 2\pi \int_0^H d\varsigma \int_0^R \rho u^0(\rho,\varsigma) G(r,\rho,z,\varsigma,t) d\rho +$$

$$2\pi a^2 R \int_0^t d\tau \int_0^H \gamma_1(\varsigma,\tau) G(r,R,z,\varsigma,t-\tau) d\varsigma - \qquad (2.3)$$

$$2\pi a^2 \int_0^t d\tau \int_0^R \rho \gamma_2(\rho,\tau) G(r,\rho,z,0,t-\tau) d\rho -$$

$$2\pi a^2 \int_0^t d\tau \int_0^R \rho \gamma_3(\rho,\tau) G(r,z,\rho,H,t-\tau) d\rho +$$

$$2\pi \int_0^t d\tau \int_0^H d\varsigma \int_0^R \rho f(\rho,\varsigma,\tau) G(r,z,\rho,H,t-\tau) d\rho$$

Now it is the right time to use the additional condition (1.8) in (2.2) and we have reduced the searched function $W(r,z,t)$ to first kind Fredholm type integral equation:

$$\int_0^T d\tau \int_0^H d\varsigma \int_0^R \rho G(r,z,\rho,\varsigma,T-\tau) W(\rho,\varsigma,\tau) d\rho = \frac{\Gamma(r,z,T) - U(r,z)}{2\pi\tau_r}.$$

The solving of this integral equation is ill-posed problem, but regardless of this, it can be solved, e.g. by Tichonov regularization method. Let us denote the regularized solution by $\tilde{W}(r,z,t)$. Then the approximate (regularized) solution $\tilde{u}(r,z,t)$ follows from the formula (2.2):

$$\tilde{u}(r,z,t) = \Gamma(r,z,t) - 2\pi\tau_r \int_0^t d\tau \int_0^H d\varsigma \int_0^R \rho G(r,\rho,z,\varsigma,t-\tau) \tilde{W}(\rho,\varsigma,\tau) d\rho. \qquad (2.4)$$

From here we have following expression for the first time derivative of the solution:

$$\frac{\partial \tilde{u}(r,z,t)}{\partial t} = \frac{\partial \Gamma(r,z,t)}{\partial t} - 2\pi\tau_r \int_0^H d\varsigma \int_0^R \rho G(r,\rho,z,\varsigma,+0)\tilde{W}(\rho,\varsigma,t)d\rho -$$

$$2\pi\tau_r \int_0^t d\tau \int_0^H d\varsigma \int_0^R \rho \frac{\partial G(r,\rho,z,\varsigma,t-\tau)}{\partial t} \tilde{W}(\rho,\varsigma,\tau)d\rho.$$

The well known filtration property of the Green function allows us to rewrite the last equation in the form:

$$\frac{\partial \tilde{u}(r,z,t)}{\partial t} = \frac{\partial \Gamma(r,z,t)}{\partial t} - 2\pi\tau_r \tilde{W}(r,z,t) -$$

$$2\pi\tau_r \int_0^t d\tau \int_0^H d\varsigma \int_0^R \rho \frac{\partial G(r,\rho,z,\varsigma,t-\tau)}{\partial t} \tilde{W}(\rho,\varsigma,\tau)d\rho. \tag{2.5}$$

The unknown function $V_0(r,z)$ can now be obtained by passage to the limit $t \to +0$ in the equation (2.5). So the solution of the time inverse problem looks as follows:

$$V_0(r,z) = \frac{\partial \Gamma(r,z,+0)}{\partial t} - \tau_r \tilde{W}(r,z,+0). \tag{2.6}$$

Conclusions

We have obtained the solution of the time inverse problem in closed form (2.6).

References

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Authors

Dr.-Habil. Math. Buikis, Andris
Dr.-Math. Guseinov, Sharif
M.Sc.-Math. Buike, Margarita
Institute of Mathematics of
Academy of Sciences and University of Latvia
Akademijas l
LV-1524 Riga, Latvia
E-mail: buikis@latnet.lv, sh.e.guseinov@inbox.lv