

Mathematical analysis of the oscillations of a liquid metal drop submitted to low frequency magnetic fields

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Abstract

We investigate the free-surface deformation of a circular liquid-metal pool under the influence of a vertical low frequency alternating magnetic field. We develop a heuristic mathematical model based on a Lagrangian approach. The system reduces to a set of two differential equations governing the vertical deformation as well as the amplitude of a single horizontal mode. The theory is applied in the case of an elongated drop. It is shown that the horizontal deformation is governed by a Mathieu-type equation giving birth to a parametric instability. Thanks to the model we retrieve qualitatively the main phenomena observed in previous experiments.

1. Introduction

When a pool of liquid metal is placed in an alternating magnetic field, the induced electric currents interact with the applied magnetic field to create electromagnetic body forces known as Lorenz forces. These Lorenz forces are composed of a mean (time averaged) part and an oscillating part, which has frequency twice that of the applied field. Both are responsible for bulk motion and free-surface deformation. These phenomena are encountered in many metallurgical applications, such as electromagnetic stirring, levitation and quasi-levitation of liquid metals.

At very low frequencies, the mean value of the Lorenz force is negligible compared to the oscillating part [1]. Also, while the oscillating part is irrotational and does not directly drive any fluid motion [2], it does modify the pressure field. In this way it may be indirectly responsible for fluid motion. Previous experimental work [2] has shown that a uniform vertical low frequency magnetic field (of the order of a few hertz) can generate different types of waves at the surface of a 200mm diameter mercury pool. Galpin et al. [3] performed a theoretical stability analysis and showed that both forced axisymmetric and non-symmetric waves (resulting from parametric instabilities) were excited. In an experimental investigation the free surface instabilities in an initially cylindrical drop of liquid mercury, Fautrelle et al. [4] observe both axisymmetric and azimuthal waves corresponding to parametric instabilities driven by a low frequency magnetic field.

The present paper is an attempt to develop a heuristic mathematical model devoted to the analysis of the phenomena observed in previous theoretical and experimental investigations [4-5]. We present a theoretical analysis in which we extend the earlier model (A.D. Sneyd & Fautrelle (2003)) to our geometry and superimpose infinitesimal azimuthal waves on the edges. The results are compared with an experimental investigation of the horizontal free surface instabilities which arise in an elongated pool of mercury placed in a uniform vertical low frequency magnetic field. Details of the derivation will not be provided here. The reader is referred to Spragg's thesis [6].

2. The mathematical model

As a first attempt to model the strip we consider a rectangular strip, resting on a substrate at $z = 0$, extending to infinity along both directions of the y axis (see Fig. 1). The strip is submitted to a uniform vertical alternating magnetic field

$$\mathbf{B} = B \sin(\omega t) \hat{\mathbf{z}}, \quad (1)$$

B , $f = \omega/2\pi$ and $\hat{\mathbf{z}}$ denoting respectively the typical magnetic field strength, its frequency and the vertical unit vector. We assume the field frequency f is sufficiently low so that the fluid motion has little influence on the applied field. We assume the average pool width x_0 is much larger than the pool height h and thus neglect the curvature of the edge, which we take to be vertical. Ignoring the forced standing waves observed in experiments on the upper surface for analytic simplicity, we consider only the first type of motion which is consistent in assuming that the upper surface $z = h(t)$ remains plane. In the following two sections we proceed to develop models to describe the two types of edge instability noticed in the experiments [6], which we have named the V (varicose) and S (sinuous) modes. In the following section we give a summary of the derivation of the differential equations which govern the S modes. The V modes are treated in the same way.

2.1. S modes

The characteristic property of the S modes is that the pool horizontal deformation is symmetric about the y axis, as shown in Fig. 1. We model the instability of the mode by allowing a small sinusoidal displacement of the edge $c(y,t)$ so that its equation is

$$c(y,t) = x_0(t) + \varepsilon b(t) \cos(ky), \quad (2)$$

where ε is a small parameter such that $\varepsilon b(t) \ll x_0(t)$ and $b(t)$ is a function to be determined. Although, in the experiments we observed that the waves on the edge are of length $l_e = \pi/k$, we use $l = 2\pi/k$ to denote the wavelength of the edge in the model (and thus model the equivalent of 2 waves) to simplify the algebra. It follows from symmetry that the volume V enclosed between one wavelength l of the edge and the y axis is constant, and given by

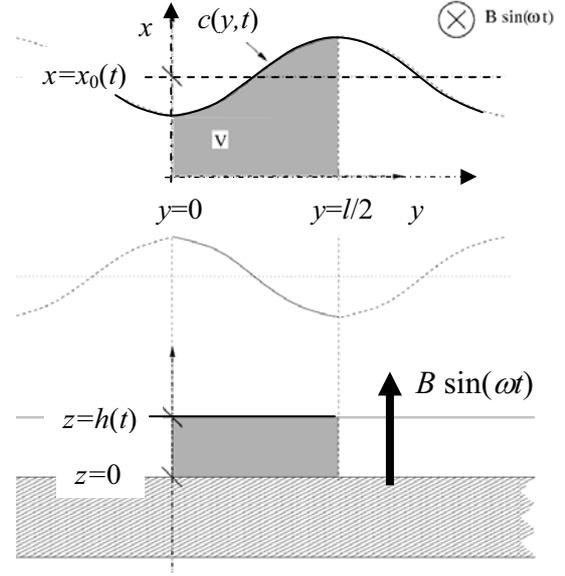
$$V = hx_0l. \quad (3)$$

From which follows that

$$\dot{x}_0 = -\frac{V\dot{h}}{h^2l}. \quad (4)$$

We assume that the fluid motion is inviscid and irrotational. We consider symmetric (about the y axis) oscillations in $h(t)$ and $x(t)$ - the strip broadening and becoming thinner or narrowing and becoming higher in accordance with Equation (4), and azimuthal oscillations in $b(t)$ induced by the oscillating magnetic field. Note the following analysis is only valid for small perturbations of the straight edge.

Fig. 1 Sketch of the geometry of the domain used in the theoretical analysis. Top figure: view from above. Bottom figure: side view. The figure corresponds to a S-mode.



We use a Lagrangian analytic method similar to that of Sneyd et al. [4] defining a Lagrangian function involving the kinetic, surface, gravitational and magnetic potential energy of the strip. We then proceed to derive the final evolution equations of the system which describe the coupled oscillations in both height h and edge perturbation b .

2.2. Lagrange equations

The Lagrangian approach is possible since in the present case all the body forces derive from a scalar potential. The Lagrangian L of the system can be written in the form

$$L = T - (E_m + E_s + E_g), \quad (5)$$

where T is the kinetic energy and E_m , E_s , E_g respectively denote the magnetic, interfacial and gravitational energies. E_m , E_s and E_g can be expressed in term of the two parameters $h(t)$ and $b(t)$.

Without entering into the details, we may illustrate how for example the kinetic energy is calculated in the present model. Owing to the hypotheses used, the velocity field is irrotational and is thus governed by a harmonic scalar potential ϕ , which must satisfy the boundary conditions especially at the edge $c(y, t)$. The potential may be expressed in term of an expansion, and to the order ε^2 we may write:

$$\phi = \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \dots, \quad (6)$$

with $\phi_0 = \frac{\dot{h}}{2h}(z^2 - x^2)$, $\phi_1 = C_1 \cos(ky) \cosh(kx)$ and $\phi_2 = C_2 \cos(2ky) \cosh(2kx)$,

$$\text{and } C_1 = \frac{\dot{b} + \dot{h}bh^{-1}}{k \sinh(kx_0)}, \quad C_2 = -b(\dot{b} + \dot{h}bh^{-1}) \frac{\coth(kx_0)}{2 \sinh(2kx_0)}.$$

Then the total kinetic energy is obtained by replacing the expression (6) of ϕ in the following integral:

$$T = \frac{1}{2} \rho \int \nabla \phi \cdot \nabla \phi \, dv = Q_0 \dot{h}'^2 + \varepsilon^2 (Q_1 \dot{h}'^2 + 2R \dot{h}' \dot{b}' + S \dot{b}'^2), \quad (7)$$

where h' and b' are non-dimensional variables such that h' / h_0 and b' / h_0 with $t' = t / \sqrt{h_0 / g}$, g being the gravity. The non-dimensional coefficients (Q_0 etc.) are given in [6].

The non-dimensional expressions of the surface energy, obtained from the total area of the drop, as well as the gravitational energy are:

$$E_S = \frac{1}{2h'} + \frac{1}{4} h' \left(l' + \frac{\pi}{2} \varepsilon^2 k' b'^2 \right), \quad E_g = \frac{1}{2} h'. \quad (8)$$

As far as the magnetic energy is concerned, it has been demonstrated that in the low-frequency limit there existed a scalar potential of the electromagnetic forces [7]. Thus the magnetic energy may be readily obtained:

$$E_m = \frac{1}{12} V' \omega'^2 N \sin(2\omega t) \left[x_0'^2 - 3 \varepsilon^2 b'^2 (1 - k' x_0' \tanh(k' x_0')) \right], \quad (9)$$

N being the non-dimensional interaction parameter such that

$$N = \frac{\sigma B^2}{\rho \omega},$$

which may be interpreted as the ratio between the electromagnetic forces and inertia.

Finally, the Lagrange equations are obtained from their formal expression:

$$\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad \text{with } q_1 = h(t) \text{ and } q_2 = b(t). \quad (10)$$

Replacing the expression of the different energies in the Lagrangian and using (10) yields two second order ordinary differential equations governing $h(t)$ and $b(t)$. The detailed expressions of the equations are provided in [6]. The equations were solved numerically, and the stability boundary was determined by a bisection method.

3. Results and discussion

From the analysis of Equations (10), it is found that $h(t)$ is governed by a non-linear forced-oscillation equation, whilst the perturbation $b(t)$ is ruled by a Mathieu-type equation. The

vertical motion is directly driven by the alternating part of the Lorentz forces. The lateral deformations are the result of an instability. The behaviour of $b(t)$ is of parametric type. The edge amplitude $b(t)$ is non-zero only beyond a critical value of the magnetic field. Beyond that threshold $b(t)$ oscillates at a frequency f (half of the electromagnetic forces). The corresponding stability diagram is shown in Fig. 2 for the two types of modes. Note that the eigenfrequencies corresponding to the S and V-modes are quite close.

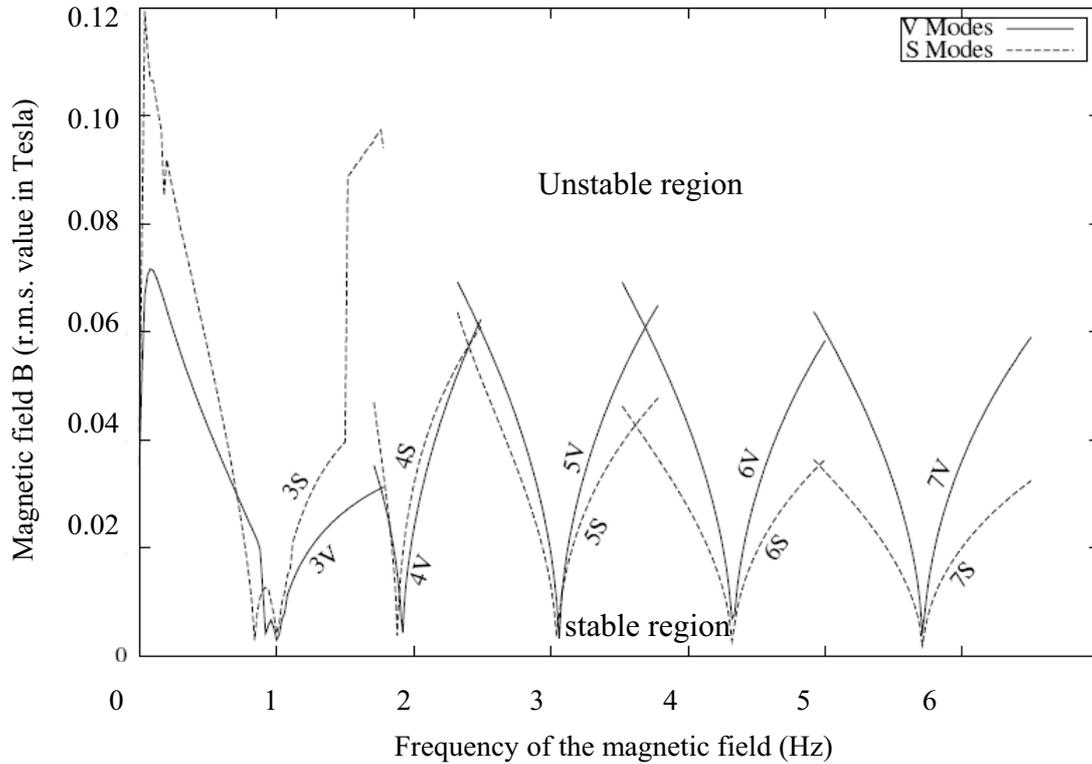


Fig. 2 Computed stability diagram for the non-symmetric perturbation $b(t)$. The stability curves were calculated for both the V- and the S-modes (respectively full and dashed lines)

For each mode, the stability boundary consists of a tongue located near the eigenfrequency of the mode. Those features are very similar to those encountered in the parametric instability.

Experiments with liquid gallium were carried out [6]. The estimated dimensions of the pool at rest are: length 84mm, width 34mm and height 6.4mm. The details of the experiments may be found in (10). It was observed various regime according to the magnetic field amplitude:

- (i) for weak magnetic field strength forced oscillations symmetric with respect to the y-axis are first observed. The oscillation period is $2f$.
- (ii) for a certain threshold non-symmetric horizontal waves appear having an oscillation frequency equal to f . Fig. 3 illustrates the various wave patterns.
- (iii) for large magnetic field values unstructured free surface patterns are generated.

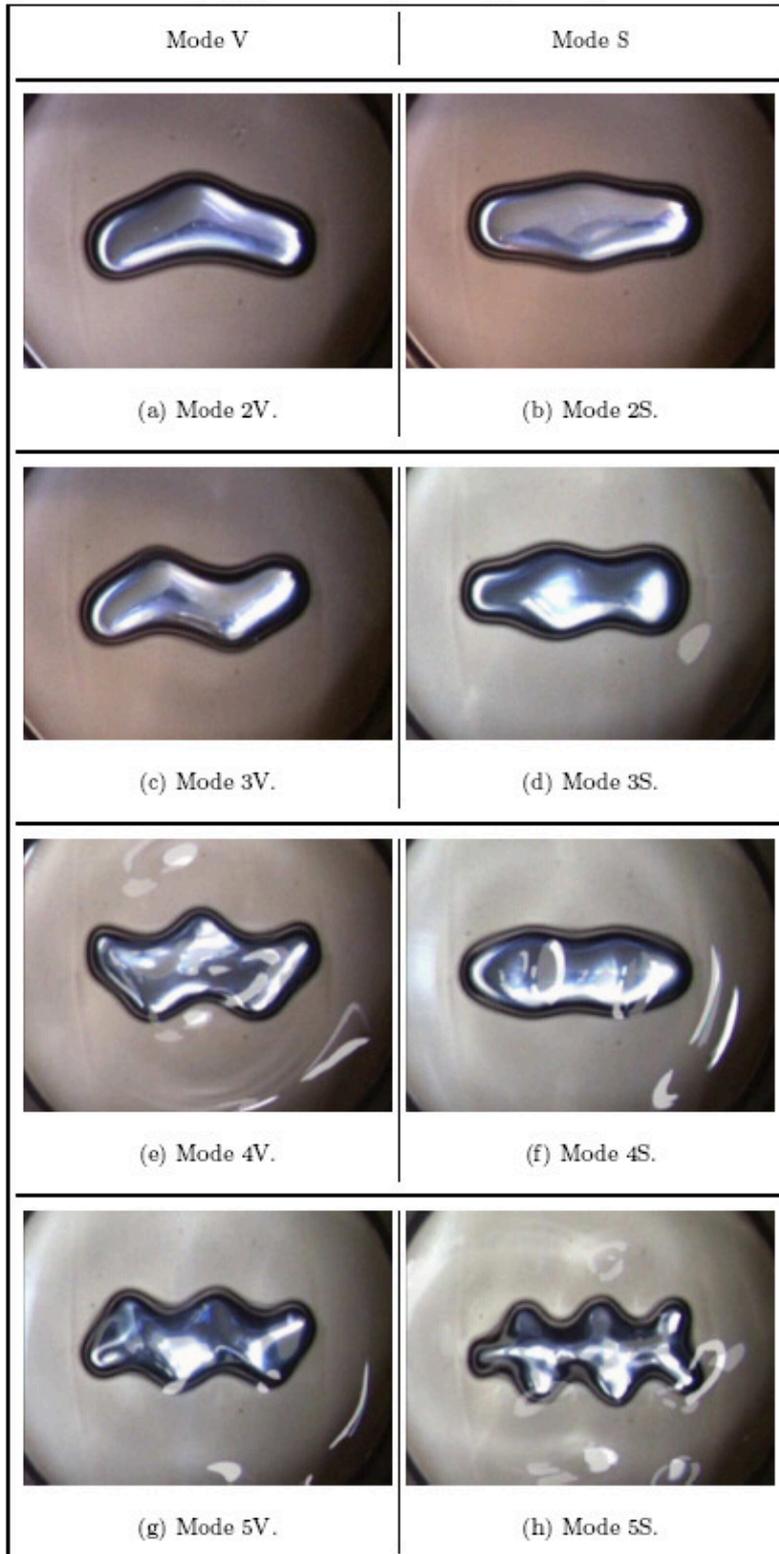


Fig. 3 View of the experimental free surface deformations of a liquid gallium drop for various coil current frequency and magnetic field amplitude in the parametric regime [6]. The frequency is varying from 1.7Hz for the mode 2 to 4.0Hz for the mode 5. The magnetic field amplitude is of order of 0.1T.

4. Conclusions

It may be concluded that the present model is heuristic, but is able to capture the main features observed from the experiments, namely:

- the existence of forced waves,
- the appearance of an instability of parametric type giving birth to horizontal wave motion.

However, a detailed comparison between the experimental stability diagram and the theoretical shows that there is still a discrepancy concerning the slope of the tongue branches. Further investigations are being carried out to explain the discrepancy.

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