Exact Analytical 3-D Steady-State Solution for Cylindrical Fin

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Abstract: - In this paper we construct two exact analytical three-dimensional solutions for cylindrical wall and fin. We assume that the heat transfer process in the wall and the fin is stationary.

Key-Words: - steady-state, three-dimensional, heat exchange, cylindrical fin, analytical solution.

1 Introduction

Usually mathematical modeling of systems with extended surfaces is realized by one dimensional steady-state assumptions [1]-[4]. In our previous papers we have constructed two and three dimensional analytical approximate [5]-[8] solutions. In paper [9] was constructed exact 2-D solution for rectangular fin. Here we construct the solution in different from [9] way. This way gives more suitable form of the solution in the form of Fredholm integral equation. We reduce exact 3-D problem to two dimensional and obtain exact analytical two-dimensional solution by the Green function method.

2 Mathematical Formulation of 3-D Problem and Exact its Reduction to Non-homogeneous 2-D Problem

We will start with accurate three-dimensional formulation of steady-state problem for system of cylindrical wall and fin. The one element of the wall (base) is placed in the domain $\left\{\tilde{r}\in[R_0,R_1],\tilde{z}\in[0,H],\varphi\in[0,\Phi]\right\}$ and we describe temperature field $\tilde{V}_0(\tilde{r},\tilde{z},\varphi)$ in the wall with the equation:

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\frac{\partial\tilde{V}_0}{\partial\tilde{r}}\right)+\frac{\partial^2\tilde{V}_0}{\partial\tilde{z}^2}+\frac{1}{\tilde{r}^2}\frac{\partial\tilde{V}_0}{\partial\varphi^2}=0. \qquad (1)$$

The cylindrical fin of length L occupies the domain $\left\{\tilde{r}\in[R_1,R_2],\tilde{z}\in[0,H_0],\varphi\in[0,\Phi]\right\}$ and the temperature field $\tilde{V}(\tilde{r},\tilde{z},\varphi)$ fulfills the equation:

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\frac{\partial\tilde{V}}{\partial\tilde{r}}\right)+\frac{\partial^2\tilde{V}}{\partial\tilde{z}^2}+\frac{1}{\tilde{r}^2}\frac{\partial\tilde{V}}{\partial\varphi^2}=0 \qquad (2)$$

And we have following boundary conditions in φ direction (others needed boundary conditions will be added in non-dimensional form in next sub-section):

$$\left.\frac{\partial\tilde{V}_0}{\partial\varphi}\right|_{\varphi=0}=\tilde{q}_0^0(\tilde{r},\tilde{z}),\left.\frac{\partial\tilde{V}_0}{\partial\varphi}\right|_{\varphi=\Phi}=\tilde{q}_1^0(\tilde{r},\tilde{z}),$$
$$\left.\frac{\partial\tilde{V}}{\partial\varphi}\right|_{\varphi=0}=\tilde{q}_0(\tilde{r},\tilde{z}),\left.\frac{\partial\tilde{V}}{\partial\varphi}\right|_{\varphi=\Phi}=\tilde{q}_1(\tilde{r},\tilde{z}). \qquad (3)$$

Introducing following average integral values for argument φ we can reduce equations (1) and (2) from 3-D to 2-D:

$$\tilde{U}(\tilde{r},\tilde{z})=\frac{1}{\Phi}\int_0^\Phi\tilde{V}(\tilde{r},\tilde{z},\varphi)d\varphi,$$
$$\tilde{U}_0(\tilde{r},\tilde{z})=\frac{1}{\Phi}\int_0^\Phi\tilde{V}_0(\tilde{r},\tilde{z},\varphi)d\varphi. \qquad (4)$$

Integration the equation (1) for the wall over $\varphi\in[0,\Phi]$ gives following equation (exact consequence of 3-D partial differential equation (1)):

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\frac{\partial\tilde{U}_0}{\partial\tilde{r}}\right)+\frac{\partial^2\tilde{U}_0}{\partial\tilde{z}^2}+$$
$$\frac{1}{\tilde{r}^2\Phi}\left(\left.\frac{\partial\tilde{V}_0}{\partial\varphi}\right|_{\varphi=\Phi}-\left.\frac{\partial\tilde{V}_0}{\partial\varphi}\right|_{\varphi=0}\right)=0.$$

The first pair of boundary conditions (3) allows rewrites the last equality in form of two-dimensional non-homogeneous equation:

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\frac{\partial\tilde{U}_0}{\partial\tilde{r}}\right)+\frac{\partial^2\tilde{U}_0}{\partial\tilde{z}^2}+\tilde{Q}_0(\tilde{r},\tilde{z})=0,$$
$$\tilde{Q}_0(\tilde{r},\tilde{z})=\frac{\tilde{q}_1^0(\tilde{r},\tilde{z})-\tilde{q}_0^0(\tilde{r},\tilde{z})}{\tilde{r}^2\Phi}. \quad (5)$$

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\frac{\partial\tilde{U}}{\partial\tilde{r}}\right)+\frac{\partial^2\tilde{U}}{\partial\tilde{z}^2}+\tilde{Q}(\tilde{r},\tilde{z})=0,$$
$$\tilde{Q}(\tilde{r},\tilde{z})=\frac{\tilde{q}_1(\tilde{r},\tilde{z})-\tilde{q}_0(\tilde{r},\tilde{z})}{\tilde{r}^2\Phi}. \quad (6)$$

2.1 Dimensionless Temperature Field in the Wall

We will use following dimensionless arguments, parameters to transform our problem to dimensionless problem:

$$r=\frac{\tilde{r}}{H}, z=\frac{\tilde{z}}{H}, \rho_0=\frac{R_0}{H}, \rho_1=\frac{R_1}{H}, \rho_2=\frac{R_2}{H},$$
$$b=\frac{H_0}{H}, \beta_0=\frac{hH}{k_0}, \beta=\frac{hH}{k}, \beta_0^0=\frac{h_0H}{k_0},$$
$$\gamma_0=\frac{\beta_0}{\rho_1}, \gamma=\frac{\beta}{\rho_1}, \gamma_0^0=\frac{\beta_0^0}{\rho_0}.$$

And temperatures:

$$U(r,z)=\frac{\tilde{U}(r,z)-T_a}{T_b-T_a}, U_0(r,z)=\frac{\tilde{U}_0(r,z)-T_a}{T_b-T_a}.$$

Here $k(k_0)$ - heat conductivity coefficient for the fin (wall), $h(h_0)$ - heat exchange coefficient for the fin (wall), H_0 - width (thickness) of the fin, L - length of the fin, H - thickness of the wall, T_b - the surrounding temperature on the left (hot) side (the heat source side) of the wall, T_a - the surrounding temperature on the right (cold - the heat sink side) side of the wall and the fin. One element of the wall (base) placed in the domain now is $\{r\in[\rho_0,\rho], z\in[0,1]\}$ and we can describe the dimensionless temperature field $U_0(r,z)$ in the wall with the equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_0}{\partial r}\right)+\frac{\partial^2U_0}{\partial z^2}+Q_0(r,z)=0. \quad (7)$$

We add needed boundary conditions as follow:

$$\frac{\partial U_0}{\partial r}+\gamma_0^0\left(1-U_0\right)=0, r=\rho_0, z\in[0,1], \quad (8)$$

$$\frac{\partial U_0}{\partial r}+\gamma_0U_0=0, r=\rho_1, z\in[b,1]. \quad (9)$$

And in contradiction with our previous papers we assume general non-homogeneous boundary conditions on top and the bottom of the wall (the same generalization will be assumed for the fin):

$$\left.\frac{\partial U_0}{\partial z}\right|_{z=0}=q_0(r), r\in[\rho_0,\rho_1],$$
$$\left.\left(\frac{\partial U_0}{\partial z}+\beta_0U_0\right)\right|_{z=1}=q_1(r). \quad (10)$$

We assume the conjugations conditions on the surface between the wall and the fin as ideal thermal contact - there is no contact resistance:

$$U_0\big|_{\rho_1-0}=U\big|_{\rho_1+0}, \gamma\left.\frac{\partial U_0}{\partial r}\right|_{\rho_1-0}=\gamma_0\left.\frac{\partial U}{\partial r}\right|_{\rho_1+0}. \quad (11)$$

2.2 The Temperature Field in the Fin

The cylindrical fin of length l occupies the domain $\{r\in[\rho_1,\rho_2], z\in[0,b]\}$ and the temperature field $U(r,z)$ fulfills the equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{\partial^2U}{\partial z^2}+Q(r,z)=0. \quad (12)$$

We have following boundary conditions for the fin:

$$\frac{\partial U}{\partial z}+\beta U=0, r\in[\rho_1,\rho_2], z=b, \quad (13)$$
$$\frac{\partial U}{\partial r}+\gamma U=0, r=\rho_2, z\in[0,b]. \quad (14)$$

And non-homogeneous boundary conditions:

$$\left.\frac{\partial U}{\partial z}\right|_{z=0}=q(r), r\in[\rho_1,\rho_2]. \quad (15)$$

3 Exact Solution of 2-D Problem

To clearer explain the main idea we will start with the case of homogeneous equations for the cylindrical wall and fin and for simplicity in this section we assume additionally homogeneous boundary conditions (10), (15). The general case (with non-homogeneous differential equations and non- homogeneous boundary conditions) will be considered in next section.

3.1 The Separate 2-D Problem for the Wall

We rewrite the boundary condition (9) together with the conjugation conditions (11) in following common form:

$$\left(\frac{\partial U_0}{\partial r}+\gamma_0 U_0\right)\Bigg|_{r=\rho_1-0}=\begin{cases}\gamma_0 F_0(z), 0<z<b,\\ 0 \qquad\qquad ,b<z<1,\end{cases}$$
$$F_0(z)=\left(\frac{1}{\gamma}\frac{\partial U}{\partial r}+U\right)\Bigg|_{r=\rho_1+0}. \tag{16}$$

The solution for the wall can be written in well known form by means of Green function, see, e.g. [10]-[12]:

$$U_0(r,z)=$$
$$2\pi\gamma_0^0\rho_0\int_0^1 G_0(r,z,\rho_0,\eta_0)d\eta_0+2\pi\gamma_0\rho_1\times$$
$$\int_0^b\left(\frac{1}{\gamma}\frac{\partial U}{\partial \xi_0}+U\right)\Bigg|_{\xi_0=\rho_1+0} G_0(r,z,\rho_1,\eta_0)d\eta_0.$$

$$V_0(r,z)=2\pi\gamma_0^0\delta_0\int_0^1 G_0(r,z,\delta_0,\eta_0)d\eta+2\pi\gamma_0\delta_0\times\int_0^b\left(\frac{1}{\gamma}\frac{\partial V}{\partial \xi_0}+V\right)\Bigg|_{\xi_0=\delta+0} G_0(r,z,\delta,\eta_0)d\eta_0 \tag{17}$$

The two-dimensional Green function has following form:

$$G_0(r,z,\xi,\eta)=G_0^{(r)}(r,\xi)G_0^{(z)}(z,\eta), \tag{18}$$

The one-dimensional Green functions have such representations:

$$G_0^{(r)}(r,\xi)=\frac{\pi}{4}\sum_{k=1}^{\infty}\frac{\lambda_k^2}{B_k}H_k(r)H_k(\xi)\times$$
$$\left[\gamma_0 J_0(\lambda_k\rho_1)-\lambda_k J_1(\lambda_k\rho_1)\right]^2, \tag{19}$$
$$B_k=\left(\lambda_k^2+(\gamma_0)^2\right)\left[\gamma_0^0 J_0(\lambda_k\rho_0)+\lambda_k J_1(\lambda_k\rho_0)\right]^2$$
$$-\left(\lambda_k^2+(\gamma_0^0)^2\right)\left[\gamma_0 J_0(\lambda_k\rho_1)-\lambda_k J_1(\lambda_k\rho_1)\right]^2,$$
$$H_k(r)=\left[\gamma_0^0 Y_0(\lambda_k\rho_0)+\lambda_k Y_1(\lambda_k\rho_0)\right]\times$$
$$J_0(\lambda_k r)-\left[\gamma_0^0 J_0(\lambda_k\rho_0)+\lambda_k J_1(\lambda_k\rho_0)\right]Y_0(\lambda_k r);$$

$$G_0^{(z)}(z,\eta)=2\sum_{l=1}^{\infty}\varphi_l(z)\varphi_l(\eta), \tag{20}$$
$$\varphi_l(z)=\cos(\mu_l z),\ \mu_l=(n+1/2)\pi.$$

Here $\lambda_k(\mu_l)$ are positive roots of following transcendental equations:

$$\left[\gamma_0^0 J_0(\lambda\rho_0)+\lambda J_1(\lambda\rho_0)\right]\times$$
$$\left[\gamma_0 Y_0(\lambda\rho_1)-\lambda Y_1(\lambda\rho_1)\right]-$$
$$\left[\gamma_0 J_0(\lambda\rho_1)-\lambda J_1(\lambda\rho_1)\right]\times$$
$$\left[\gamma_0^0 Y_0(\lambda\rho_0)+\lambda Y_1(\lambda\rho_0)\right]=0.$$

Representation (17) in notations (16) can be rewritten in shorter form (with known function $\Phi_0(r,z)$):

$$U_0(r,z)=\Phi_0(r,z)+$$
$$2\pi\gamma_0\rho_1\int_0^b F_0(\eta_0)G_0(r,z,\rho_1,\eta_0)d\eta_0. \tag{21}$$

$$V_0(r,z)=\Phi_0(r,z)+2\pi\gamma_0\delta_0\int_0^b F_0(\eta_0)G_0(r,z,\delta,\eta_0)d\eta_0$$

This function is given by expression:

$$\Phi_0(r,z)=2\pi\gamma_0^0\rho_0\int_0^1 G_0(r,z,\rho_0,\eta_0)d\eta_0.$$

The representation (21) is not the solution because of unknown function $F_0(\eta)$.

3.2 The Separate 2-D Problem for the Fin

$$\widetilde{V}(r,z)=-2\pi\delta\times\int_0^b\left(\frac{1}{\gamma_0}\frac{\partial\widetilde{V}_0}{\partial\xi_0}-\widetilde{V}_0\right)\Bigg|_{\xi_0=\delta-0} G(r,z,\delta,\eta)d\eta$$

We rewrite the conjugations conditions (11) in following form:

$$\left(\frac{\partial U}{\partial r}-\gamma U\right)\Bigg|_{r=\rho_1+0}=\gamma F(z),$$
$$F(z)=\left(\frac{1}{\gamma_0}\frac{\partial U_0}{\partial r}-U_0\right)\Bigg|_{r=\rho_1-0}. \tag{22}$$

Similar as in case of wall the solution for the fin can be represent in following form:

$$U(r,z)=-2\pi\gamma\rho_1\times$$
$$\int_0^b\left(\frac{1}{\gamma_0}\frac{\partial U_0}{\partial \xi_0}-U_0\right)\Bigg|_{\xi_0=\rho_1-0} G(r,z,\rho_1,\eta)d\eta. \tag{23}$$
$$V(r,z)=-2\pi\delta\int_0^b F(\eta)G(r,z,\delta,\eta)d\eta$$

The shorter form looks as follow:

$$U(r,z)=-2\pi\gamma\rho_1\int_0^b F(\eta)G(r,z,\rho_1,\eta)d\eta. \tag{24}$$

The second Green function with small modifications has a same form as expression (18):

$$G^{(r)}(r,\xi)=\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{\lambda_n^2}{B_n}H_n(r)H_n(\xi)\times$$
$$\left[\gamma J_0(\lambda_n\rho_2)-\lambda_n J_1(\lambda_n\rho_2)\right]^2,$$
$$B_n=\left(\lambda_n^2+(\gamma)^2\right)\left[\gamma J_0(\lambda_n\rho_1)+\lambda_n J_1(\lambda_n\rho_1)\right]^2$$
$$-\left(\lambda_n^2+(\gamma)^2\right)\left[\gamma J_0(\lambda_n\rho_2)-\lambda_n J_1(\lambda_n\rho_2)\right]^2,$$
$$H_n(r)=\left[\gamma Y_0(\lambda_n\rho_1)+\lambda_n Y_1(\lambda_n\rho_1)\right]J_0(\lambda_n r)$$
$$-\left[\gamma J_0(\lambda_n\rho_1)+\lambda_n J_1(\lambda_n\rho_1)\right]Y_0(\lambda_n r);$$
$$G_0^{(z)}(z,\eta)=\sum_{m=1}^{\infty}\frac{\varphi_m(z)\varphi_m(\eta)}{\|\varphi_m\|^2},$$
$$\|\varphi_m\|^2=\gamma+\frac{b\left(\mu_m^2+\gamma^2\right)}{2},$$
$$\varphi_l(z)=\mu_m\cos(\mu_m z)+\gamma\sin\left(\mu_m z\right).$$

Here $\lambda_n(\mu_m)$ are positive roots of following transcendental equations:

$$\left[\gamma J_0(\lambda\rho_1)+\lambda J_1(\lambda\rho_1)\right]\left[\gamma Y_0(\lambda\rho_2)-\lambda Y_1(\lambda\rho_2)\right]-$$
$$\left[\gamma J_0(\lambda\rho_2)-\lambda J_1(\lambda\rho_2)\right]\left[\gamma Y_0(\lambda\rho_1)+\lambda Y_1(\lambda\rho_1)\right]=0,$$
$$\mu=\gamma\cot\left(\mu b\right).$$

3.3 The Conjugation of Two Separate Problems

We obtain easy from representations (21), (24) following two equalities:

$$F(z)=\tilde{\Phi}_0(\rho_1,z)-$$
$$2\pi\rho_1\int_0^b F_0(\eta_0)\tilde{G}_0(\rho_1,z,\rho_1,\eta_0)d\eta_0, \tag{25}$$
$$F_0(z)=-2\pi\gamma\rho_1\int_0^b F(\eta)\tilde{G}(\rho_1,z,\rho_1,\eta)d\eta.$$

Here

$$\tilde{\Phi}_0(r,z)=\frac{1}{\gamma_0}\frac{\partial\Phi_0}{\partial r}-\Phi_0,$$
$$\tilde{G}_0(r,z,\xi,\eta)=\gamma_0 G-\frac{\partial G_0}{\partial r}, \tag{26}$$
$$\tilde{G}(r,z,\rho_1,\eta)=\frac{\partial G}{\partial r}+\gamma G.$$

The system (25) allows writing out following second kind Fredholm integral equation:

$$F(z)=\tilde{\Phi}_0(\rho_1,z)+\int_0^b F(\eta)\Gamma(z,\eta)d\eta. \tag{27}$$
$$F(z)=\tilde{\Phi}_0(\delta,z)+\int_0^b F(\eta)\Gamma(z,\eta)d\eta$$

Here

$$\Gamma(z,\eta)=\left(2\pi\rho_1\right)^2\gamma\times$$
$$\int_0^b\tilde{G}_0(\rho_1,z,\rho_1,\eta_0)\tilde{G}(\rho_1,z,\rho_1,\eta)d\eta_0. \tag{28}$$

This second kind Fredholm integral equation (27) by the given kernel $\Gamma(z,\eta)$ has exact one solution. Knowing $F(z)$ we can find from representation (24) the solution for the fin. In similar way we construct integral equation for the function $F_0(z)$ and find solution for the wall.

4 Exact Solution by Non-homogeneous Environment Temperature

Now we will consider the case of non-homogeneous equations and non-homogeneous boundary conditions.

4.1 The Statement of the Full Mathematical Problem

As the main equations for the wall and the fin we take differential equations (7), (12):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_0}{\partial r}\right)+\frac{\partial^2 U_0}{\partial z^2}+Q_0(r,z)=0,$$
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{\partial^2 U}{\partial z^2}+Q(r,z)=0. \tag{29}$$

The boundary conditions for the wall are as follow:

$$\left.\left(\frac{\partial U_0}{\partial r}-\gamma_0^0 U_0\right)\right|_{r=\rho_0}=-\gamma_0^0\vartheta_0(z),$$
$$\left.\left(\frac{\partial U_0}{\partial r}+\gamma_0 U_0\right)\right|_{r=\rho_1}=\gamma_0\vartheta(\rho_1,z),\ z\in[b,1],$$
$$\left.\frac{\partial U_0}{\partial z}\right|_{z=0}=q_0(r), r\in[\rho_0,\rho_1], \tag{30}$$
$$\left.\left(\frac{\partial U_0}{\partial z}+\beta_0 U_0\right)\right|_{z=1}=q_1(r).$$

Similar are the boundary conditions for the fin:

$$\left(\frac{\partial U}{\partial z}+\gamma U\right)\Bigg|_{z=b}=\gamma\vartheta(r,b),$$
$$r\in[\rho_1,\rho_2],$$
$$\left(\frac{\partial U}{\partial r}+\gamma U\right)\Bigg|_{r=\rho_2}=\gamma\vartheta(\rho_2,z), \tag{31}$$
$$z\in[0,b],$$
$$\frac{\partial U}{\partial z}\Bigg|_{z=0}=q(r), r\in[\rho_1,\rho_2].$$

To complete the full statement of generalized problem, we must add the conjugations conditions to equations (29)-(31):

$$U_0\big|_{\rho=\rho_1-0}=U\big|_{\rho=\rho_1+0},$$
$$\gamma\frac{\partial U_0}{\partial r}\Bigg|_{\rho=\rho_1-0}=\gamma_0\frac{\partial U}{\partial r}\Bigg|_{\rho=\rho_1+0}. \tag{32}$$

4.2 The Separate Solutions for the Wall and the Fin

In the same way as in the sub-section we introduce the notation (16). Then the solution in the wall again can be presented in the same form (21):

$$U_0(r,z)=\Phi_0(r,z)+$$
$$2\pi\gamma_0\rho_1\int_0^b F_0(\eta_0)G_0(r,z,\rho_1,\eta_0)d\eta_0. \tag{33}$$

Now the expression for the first term of right hand side has significant more complicate form:

$$\Phi_0(r,z)=$$
$$2\pi\beta_0\int_b^1\vartheta(\rho_1,\eta_0)G_0(r,z,\rho_1,\eta_0)d\eta_0+$$
$$2\pi\beta_0^0\int_0^1\vartheta_0(\eta_0)G_0(r,z,\rho_0,\eta_0)d\eta_0+$$
$$2\pi\int_{\rho_0}^{\rho_1}\xi_0 q_1(\xi_0)G_0(r,z,\xi_0,1)d\xi_0- \tag{34}$$
$$2\pi\int_{\rho_0}^{\rho_1}\xi_0 q_0(\xi_0)G_0(r,z,\xi_0,0)d\xi_0+$$
$$2\pi\int_{\rho_0}^{\rho_1}\xi_0 d\xi_0\int_0^1 Q_0(\xi_0,\eta_0)G_0(r,z,\rho_0,\eta_0)d\eta_0.$$

The solution for the wall with boundary conditions (22) in similar way as for wall can be presented in the form:

$$U(r,z)=\Phi(r,z)-$$
$$2\pi\beta\int_0^b F(\eta)G(r,z,\rho_1,\eta)d\eta. \tag{35}$$

The expression for the first term of right hand side has the form:

$$\Phi(r,z)=$$
$$2\pi\gamma\rho_2\int_0^b\vartheta(\rho_2,\eta)G(r,z,\rho_2,\eta)d\eta+$$
$$2\pi\beta\int_{\rho_1}^{\rho_2}\xi\vartheta(\xi,b)G(r,z,\xi,b)d\xi- \tag{36}$$
$$2\pi\int_{\rho_1}^{\rho_2}\xi q(\xi)G(r,z,\xi,0)d\xi+$$
$$2\pi\int_{\rho_1}^{\rho_2}\xi d\xi\int_0^b Q(\xi,\eta)G(r,z,\xi,\eta)d\eta.$$

4.3 The Junction of Solutions for the Wall and the Fin

We introduce following notations:

$$\tilde{\Phi}_0(r,z)=\frac{1}{\gamma_0}\frac{\partial\Phi_0}{\partial r}-\Phi_0,$$
$$\tilde{\Phi}(r,z)=\frac{1}{\gamma}\frac{\partial\Phi}{\partial r}+\Phi,$$
$$\tilde{G}_0(r,z,\xi,\eta)=\gamma_0 G-\frac{\partial G_0}{\partial r},$$
$$\tilde{G}(r,z,\rho_1,\eta)=\frac{\partial G}{\partial r}+\gamma G.$$

Then the representations (33) and (35) allow obtain easy following two equations:

$$F(z)=\tilde{\Phi}_0(\rho_1,z)-$$
$$2\pi\rho_1\int_0^b F_0(\eta_0)\tilde{G}_0(\rho_1,z,\rho_1,\eta_0)d\eta_0,$$
$$F_0(z)=\tilde{\Phi}(\rho_1,z)- \tag{37}$$
$$2\pi\gamma\rho_1\int_0^b F(\eta)\tilde{G}(\rho_1,z,\rho_1,\eta)d\eta.$$

From this system (37) we obtain following second kind Fredholm integral equation:

$$F(z)=\Psi_0(\rho_1,z)+\int_0^b F(\eta)\Gamma(z,\eta)d\eta. \tag{38}$$

Here the kernel of the Fredholm integral equation is given by the same formula (28):

$$\Gamma(z,\eta) = \left(2\pi\rho_1\right)^2 \gamma \times$$

$$\int_0^b \tilde{G}_0(\rho_1, z, \rho_1, \eta_0)\tilde{G}(\rho_1, z, \rho_1, \eta) d\eta_0 .$$

In its turn the first term in the right hand side has more complicate expression:

$$\Psi_0(\rho_1, z) = \tilde{\Phi}_0(\rho_1, z) - 2\pi\rho_1 \times$$

$$\int_0^b \tilde{\Phi}(\rho_1, \eta_0)\tilde{G}_0(\rho_1, z, \rho_1, \eta_0) d\eta_0 .$$

Evidently this second kind Fredholm integral equation (38) has exact one solution. Again, by known $F(z)$ the representation (35) allows find the solution for the fin. In similar way we can construct integral equation for the function $F_0(z)$ and find solution for the wall.

5 Conclusions

We have constructed two exact two-dimensional analytical solutions (in both cases: homogeneous and non-homogeneous environment) for a system with cylindrical fin when the wall and the fin consist of materials, which have different thermal properties.

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