Permeability of clustered fibre networks: modelling of unit cell

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The paper is focused on the estimation of the permeability of a clustered fibre network by variational methods. First laminar flow in ducts is considered by usage of polynomial trial functions. Next longitudinal flow through a square array is described by expansion of a velocity field in trigonometric and Laurent series. Finally, the formal scheme for the estimation of longitudinal permeability in a cluster with an irregular distribution of fibres is given. The irregular distribution is modelled by setting an individual effective radius of each fibre and then letting this fibre reach its local minima. The results obtained here form a base for future predictions of the permeability of fibre reinforcements.

Crucial for a successful manufacturing of fibre reinforced polymer composites is a perfect control of the impregnation of the fibres. Hence, there is a strong driving force for the development of methods to predict and optimise this stage of the process. Generally the flow that takes place during thermoset resin impregnation of fibres follows Darcy's law. This implies that the fibre reinforcements can be characterised by their permeability through measurements or by the usage of permeability models based on, for instance, the detailed fibre geometry. This geometry is often formed from fibre bundles consisting of a large number of fibres. In some processes, such as filament winding, the impregnation of the bundles is crucial while in other processes, such a liquid moulding, the fibre bundles themselves form a mat that is impregnated. For high strength items the mats used are formed from fibre bundles being weaved or stitched together. This often results in dual scale porosity and consequently two types of flow during impregnation: a microscale flow within the fibre bundles and a mesoscale flow between the bundles. Typical length scales for the two types of flow are < 10 μm and > 100 μm, respectively. It has been shown that the relation between the micro- and mesoflow is important for flow front phenomena such as void formation while a number of studies have indicated that the overall flow rate through the material is, to a large extent, set by the mesoflow; cf., for instance, [1-3]. This is easily understood by noticing that the volumetric flow rate per unit area is proportional to the square of the characteristic length scale in Poiseuille flow. Hence, the flow rate per unit area on the mesoscale is as a rule of thumb, one hundred times higher or more than it is on the microscale as long as the geometry on the two scales is principally the same.

A few attempts to model the permeability of repeatable cells in clustered fibre networks has been presented. For instance, Simacek and Advani [4] modelled the permeability through a weave. The problem was solved by assumption of Stokes and Darcy flow in the resin channels and the weft and warp bundles, respectively. It all boiled down to a set of two coupled two-dimensional linear PDEs. In [5] conceptual models were proposed for four representative fibre reinforcements. For one of them, a unidirectional non crimp fabric the model is simply a medium sized pore in parallel with a small pore representing the flow between and within the fibre bundles, respectively. Similar to results in [3] who also studied non crimp fabrics the model was verified

qualitatively through experiments. However, the modelling lacks quantitative conformity with experiment. One reason for this is that in-reality the fibres and the fibre bundles will not be perfectly distributed at impregnation. *Firstly*, the fibres will rarely form perfects arrangements of cylinders, cf. Figure 1. More likely are that the placement of the fibres in the plane is close to random and in the direction of the fibres, the fibres will diverge from and converge towards their nearest neighbours [3, 6, 7]. *Secondly*, similar deviations from regular packing have been observed for the fibre bundles. Although the placement of the bundles is limited to a certain volume their actual positions within this volume tend to vary. An imperfect distribution was, for example, considered in [3] by studies of a 2D-channel network. In this case the cross-section of all inter-bundle channels was assumed to be rectangular and Stokes Equations for the flow could be solved. It was furthermore assumed that the flow through the bundles is governed by Darcy law and that the Equations by Gebart [8] can be used to model the permeability. The overall strategy in [3] is promising but the cross-sectional areas of the inter-bundle channels must be better modelled and the effect on the permeability from a random distribution of the fibres should be investigated. In addition stitching and fibres crossing the inter-bundle channels have to be considered.
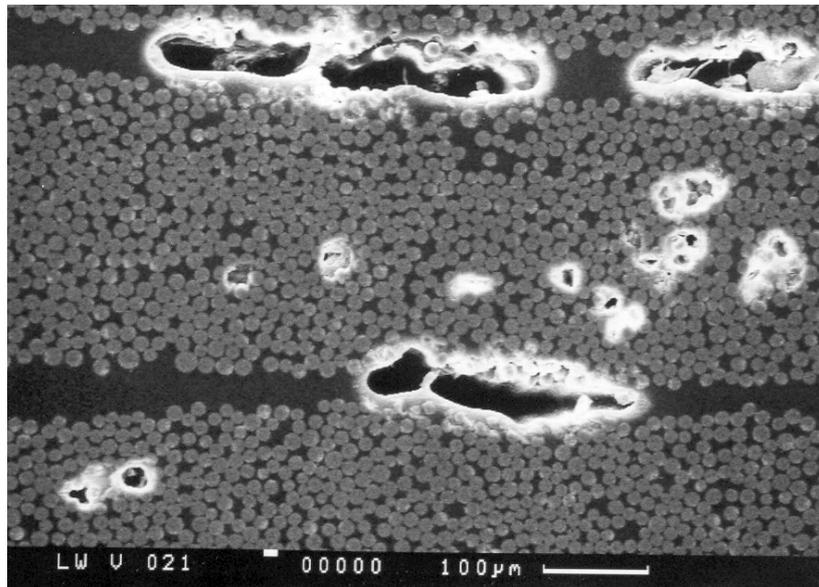

Figure 1. Polished cross-section of an RTM-moulded composite. The fibres appear as lighter grey areas on a darker grey matrix while the black areas denote voids.

To follow some of these demands the permeability of a few geometries on the meso- and microscales are studied by variation of the functional. To start with analytical expressions for flow through ducts are presented. Of special interest are ducts having cross-sectional areas close to those formed in non-crimp fabrics. Subsequently, a way to obtain a random fibre distribution is suggested and two routes to model permeability along the fibres are presented for a quadratic arrangement of fibres. Then one of these methods is used to model the permeability of a random distribution of fibres and finally some conclusions are presented.

Modelling on mesoscale

As mentioned in the introduction the flow through the space between the fibre bundles of textiles usually exceeds by some orders of magnitude the flow inside the bundles. If we consider the two-dimensional case, i.e. ducts, the space between the bundles can take up different kind of shapes ranging from rectangular to cusped [3]. A cross-section formed by two lines in parallel and two concaved boundaries covers many of these geometries. To study the flow through a tube with this cross-section we consider the stationary flow of a Newtonian, viscous and incompressible fluid. Therefore, the kinetic term is neglected and the equations become linear. For such flow the well-known permeability coefficient is proportional to the square area of the duct S. Let us use the dimensionless permeability coefficient according to:

$$K = \frac{\int u \, dS}{S^2} ,$$

(1)

where the dimensionless velocity along the duct is determined from:

$$\Delta u = -1$$

(2)

by setting a nonslip boundary conditions along the perimeter. The estimation of the permeability coefficient for relatively simple geometries can now be carried out by variation of the functional [9]:

$$\Phi_u = \int \left[\frac{(\nabla u)^2}{2} - u \right] dS .$$

(3)

The variational method allows us to find the functional dependence on the characteristic shape parameters. The aim of obtaining of the functional dependence is a subsequent statistical analysis of permeability in a bundle of fibres. To start with let us consider the case of a parallelogram. For this cross-section, the variation of polynomial trial functions gives better results than that of trigonometric ones with the same number of parameters. For example, the polynomials which satisfy the boundary conditions for a rectangle $|x| \leq a/2$, $|y| \leq b/2$ are:

$$u = \left(x^2 - \left(\frac{a}{2} \right)^2 \right) \left(y^2 - \left(\frac{b}{2} \right)^2 \right) \sum_{i,j} A_{i,j} x^{2i} y^{2j}$$

(4)

and simple analytic variation of four parameters gives the following dimensionless permeability:

$$K = \frac{7 \sin \alpha \, ab (45a^4 + 416a^2 b^2 + 45b^4)}{450(a^2 + b^2)(9a^4 + 94a^2 b^2 + 9b^4)},$$

(5)

where a and b are the sizes of the parallelogram and α is the shear angle. For the square, it gives the permeability 0.035139, while the actual value is 0.035144. The 4-parameter variation would give 0.034812. Studies made in [3] indicate that the cross-section of the ducts can typically be described by two straight lines and two parabolas:

$$|x| \leq \frac{a}{2} \left(1 + q \left(\frac{2y}{b} \right)^2 \right), \quad |y| \leq \frac{b}{2} .$$

(6)

The analysis of flow through such a duct is simplified by transformation of coordinates to a rectangle. By such a method the simplest variation of polynomial function yields the following functional dependence of permeability:

$$K = \frac{abq^2(5+q)^2}{5(3+q)^2\left(3(1+q)^2(5b^2-12a^2q)\dfrac{\arctan\sqrt{q}}{\sqrt{q}}-5b^2(3+5q)+4a^2q\left(9+17q+\dfrac{26}{5}q^2\right)\right)} \, . (7)$$

If addition parameters are introduced, it is more convenient to use a numerical approach. This is effectively performed up to the arbitrary degree with Gauss-Legandre integration of non-trivial integrals. Numerical results for the previously mentioned ducts show that with each increase of the order of polynomials the accuracy increases by one order of magnitude.

Modeling of microflow

To obtain a complete picture of the infiltration also the microflow is of highest importance. This flow may determine the amount of voids formed at the resin flow front for liquid moulding processes and is absolutely crucial for the impregnation in processes such as filament winding. To obtain the complete permeability tensor on this scale both flow along and perpendicular to the fibres must be considered [8]. However, to start with one route to describe the irregular distribution of fibres in a bundle will be outlined.

Distribution of fibres

To be able to model the permeability inside the bundles the detailed fibre distribution must be known. As seen in Fig. 1 the fibres are not distributed in a regular pattern with triangular or square symmetry, for instance. Each position is instead determined by the position of its neighbours. In order to mimic such a distribution by a computer, let us use the following model. First, we select a random radius from a Gaussian like distribution and denote this radius the effective one. The effective size, or the size the fibre will require, arises due to interlacing and inhomogeneity in real 3D structure. The apparent (the real) cross-section of the fibre is then put into the effective one. Next, a position along the horizontal axis is chosen in a random manner and the fibre is placed in the nearest minima. The main assumption introduced is that there is a perfect slip between the fibres.
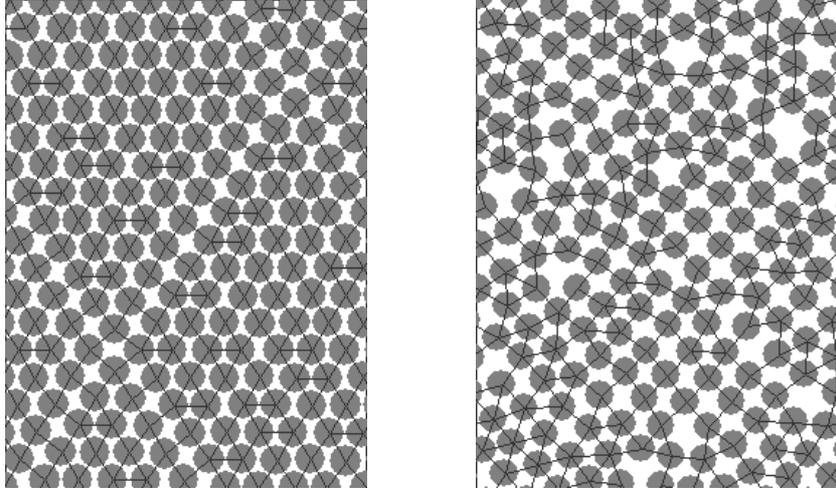
Fig. 2. The computer generated distributions of the fibre layout. The right figure has larger deviation from the constant effective radius of fibres.

If the effective radius is close to the apparent a near to regular pattern results however as the difference increases the pattern becomes irregular, see Figure 2. It now remains to find a control volume that big that we can use periodic boundary conditions along the x-axis.

Longitudinal flow inside the fibre bundles: square array

The quadratic arrangement of fibres will here be studied for two methodologies to derive the permeability, namely Trigonometric expansion and Laurent series.

For *trigonometric trial functions* a variational approach may be expressed as:

$$u = \sum_{i,j=1}^{N} A_{i,j}\phi_i(x)\phi_j(y) \tag{8}$$

with the symmetric coefficients $A_{i,j}=A_{j,i}$. The nonslip boundary condition in this case is not satisfied automatically. Therefore, the penalty functional approach should be used:

$$\Phi_u = \int\left[\frac{(\nabla u)^2}{2} - u\right]dS + \frac{\eta}{2}\int u^2 dl, \tag{9}$$

where the second integral is taken over the perimeter of fibres and the penalty function parameter η goes towards infinity. However, the parameter η must be finite for the finite set of variational parameters. The exclusion of the interior of the fibres usually provides better results for high packing, but then the trigonometric functions are not more orthogonal. The coefficients $A_{i,j}$ asymptotically depends on η as:

$$A_{i,j} \approx \frac{B_{i,j}}{\eta} + C_{i,j}. \tag{10}$$

Consider that we have found the coefficients $A_{i,j}$ from the linear system. The limit values $C_{i,j}$ can be found later from the same linear system with the following right side:

$$\int \phi_k(x)\phi_l(y)dS - \eta \sum_{i,j=1}^{N} A_{i,j} \int \phi_k(x)\phi_l(y)\phi_i(x)\phi_j(y)dl \qquad (11)$$

with previously obtained coefficients $A_{i,j}$. The Fourier spectrum in logarithmic scale for packing 0.5 is shown below.
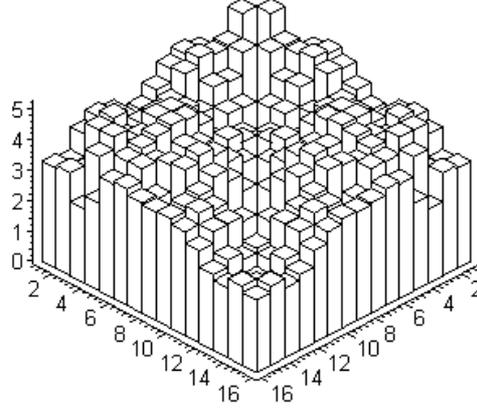

Fig. 3. Fourier spectrum in logarithmic scale for trigonometric expansion of velocity in square array of fibres at packing 0.5.

As can be seen, high harmonics are involved and expansion converges slowly. The permeability for 10 by 10 parameters is shown in Table 1 where the numerical results are copied from [11]. It is obvious that the method works only for high packing. The reason for this is that the small fibre radius involves yet higher harmonics.

We will now use *Laurent series* on the same geometry. For the squared arrangement of fibres we can chose the sub-system, which includes only 1/8 of a fibre. Let us use the polar coordinates with the centre placed in the centre of the fibre and the angular coordinate runs from 0 to $\pi/4$. The Laurent series [10]:

$$u = \frac{R^2 - r^2}{4} + B_0 \ln \frac{r}{R} + \sum_{n=1}^{N} B_n \left(\left(\frac{R}{r} \right)^{4n} - \left(\frac{r}{R} \right)^{4n} \right) \cos(4n\vartheta) \qquad (12)$$

has the given square symmetry, where r is the distance from the centre of the fibre. Moreover, it satisfies the differential equation (2) for any set of B_i and is zero at the surface of the fibre. Thus, coefficients B_i can be found so that the boundary condition at $x = l/2$ on each cell is satisfied or nearly satisfied, where l is the distance between the centres of adjacent fibres. Hence, we must minimise the functional:

$$\Phi_u = \int_0^{\frac{\pi}{4}} \left(\frac{\partial u}{\partial x} \right)^2 \frac{d\vartheta}{(\cos \upsilon)^2} \qquad (13)$$

in the class of functions given by Laurent series. The Laurent series converges rapidly, especially for small fibre volume fraction, see Table 1. As seen summation up to $N = 1$ is almost sufficient and at $N = 3$ the accuracy is about 4-digits.

Table 1. The permeability for flow along a square array of fibres where f is the fibre volume fraction.

f	Numerical result [11]	Laurent series ($N=1$)	Trigonometric function ($N=10$)
0.1	2.552	2.545	
0.15	1.182	1.178	
0.2	0.6412	0.6348	
0.25	0.3781	0.3765	
0.3	0.2350	0.2337	
0.35	0.1509	0.1500	
0.4	0.09910	0.09840	
0.45	0.06603	0.06555	0.06319
0.5	0.04442	0.04411	0.04440
0.55	0.03005	0.02987	0.02994
0.60	0.02039	0.02030	0.02037
0.65	0.01385	0.01378	0.01385
0.7	0.009413	0.009312	0.009434
0.725	0.007758	0.007623	0.007754
0.75	0.006380	0.006216	0.006302
0.775	0.005260	0.005040	0.005266

For all values in Table 1 the dimensionless permeability coefficient is obtained by division with the squared of the fibre radius:

$$\frac{\int u dS}{S_{tot} R^2},$$

(14)

where S_{tot} is the area of the cell including the fibre.

Cells of non-uniformly distributed fibres

In order to avoid boundary effects, let us consider an area with a set of fibres with periodic boundary conditions. The use of trigonometric functions in variational approach is completely similar as it was in the squared arrangement. However, it is extremely ineffective for systems involving several fibres. The superposition of the individual Laurent series gives:

$$u = \sum_{l_i} A_{l_1, l_2, \dots, l_m} \prod_i \phi_i^{l_i},$$

(15)

where ϕ is the solution for squared array of individual fibre from previous section and m the number of fibres in a periodic cell. This approach is likely to be much more effective than trigonometric functions, since the no-slip boundary condition and periodic boundary conditions are satisfied automatically. For the lowest order approximation for one coefficient, the functions A are given from:

$$A = \frac{\int \prod_i \phi_i \, dS}{\int \left(\sum_i \frac{\nabla \phi_i}{\phi_i} \prod_i \phi_i \right)^2 dS} ,$$

(16)

where the integration is performed over each periodic cell. The corresponding figure for two fibres in a periodic cell is shown in Fig. 4.
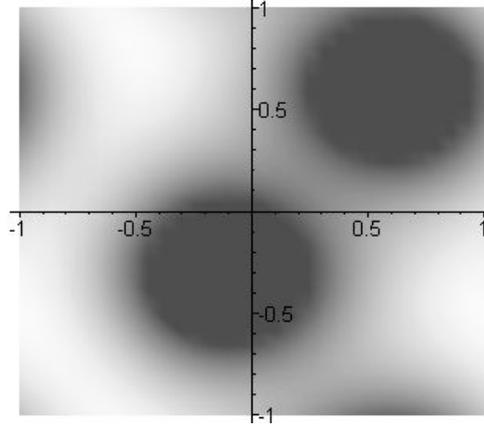

Fig. 4. Approximation of velocity distribution by individual solutions in a periodic system with two fibres.

Thus, the contribution of individual part of the opening in bundle of fibres can be evaluated. However, this approach is not effective for the system of more fibres. One way to go is to separate the entire structure of fibre distribution into small cells as indicated in Fig. 2. Because the velocity takes low values in area between closely spaced fibres, the cells can be considered as nearly independent, and the overall permeability can be estimated from the permeabilities of individual cells.

Transversal flow through fibres

Let us consider the two-dimensional isotropic case. The continuity equation and equation describing conservation of momentum should be satisfied:

$$\text{div } \mathbf{u} = 0; \text{ grad } p = \Delta \mathbf{u}$$

(17)

for incompressible fluid with rot $\mathbf{u} = 0$. The functional can now be expressed in the following way:

$$\Phi_{\mathbf{u},p} = \int \left[\frac{(\nabla u_x)^2 + (\nabla u_y)^2}{2} - \mathbf{u} \nabla p \right] dS ,$$

(18)

where the velocity components and pressure should be varied simultaneously. In some cases, it is better to use stream function χ defined as rot $\chi = \mathbf{u}$, so that continuity equation is satisfied automatically. Then, the equation $\Delta^2 \chi = 0$ is valid with corresponding functional

$$\Phi_\chi = \int \frac{(\Delta\chi)^2}{2} dS .$$ (19)

The similar expression as for the longitudinal flow (12) also for transverse flow in square array with unknown coefficients is given in [10]. These coefficients can be found by minimising the deviation from the accurate boundary conditions. As indicated in the literature, e.g. [10], the permeability in transversal direction could be several times lower than in longitudinal one. Especially critical is the case with close packed distribution of fibres.

Conclusions

The variational method can be effectively used to find the characteristic dependence of permeability of viscous incompressible fluid through ducts formed by gaps between the fibre bundles. The polynomial approach is much more appropriate than trigonometric one for the shape of ducts close to parallelogram.

Modelling of microflow is of importance to obtain the complete picture of infiltration. Letting the fibres fall one-by-one in a random place and then manoeuvre them to the local minima can generate the characteristic distribution of the fibres in the bundle. Triangular packing is more outspoken for the case when the effective fibre diameter is almost constant, while some spreading of the effective fibre diameter leads appearance of square or irregular structure.

For the periodic fibre distribution, it is much better to use the solutions of Laplace equation with given spatial symmetry, rather than to use trigonometric expansion which automatically obeys periodic boundary conditions. In the first case, the permeability coefficient can be found minimising the deviation from the exact periodic boundary condition. The expansion in Laurent series gives extremely accurate results for small packing, while trigonometric expansion works only for high packing. The product of solutions for individual fibres can estimate the velocity distribution in the system with random layout of fibres and periodic boundary conditions.

Acknowledgement

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